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An integral form of the quantized enveloping algebra of sl_2 and its completions

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Abstract

We introduce an integral form $\mathcal{U}$ of the quantized enveloping algebra of sl_2. The algebra $\mathcal{U}$ is exactly large enough so that the quasi-R-matrix is contained in a completion of $\mathcal{U} \otimes \mathcal{U}$. We study several completions of the algebra $\mathcal{U}$, and determine their centers. This study is motivated by a study of integrality properties of the quantum sl_2 invariants of links and integral homology spheres.

MSC: 17B37; 57M27

1. Introduction

The purpose of this paper is to study an integral form of the quantized enveloping algebra $U_v(sl_2)$ of the Lie algebra sl_2, and some completions of it. The motivation of this paper is a study of integrality properties of the colored Jones polynomials [12] of links and the Witten–Reshetikhin–Turaev invariant [15,13] of 3-manifolds, which we announced in [2].

Let $U = U_v(sl_2)$ denote the quantized enveloping algebra of the Lie algebra sl_2, which is defined to be the algebra over the rational function field $\mathbb{Q}(v)$ generated by the elements K, K^{-1}, E, and F, subject to the relations $KK^{-1} = K^{-1}K = 1$ and

$$KE = v^2 EK, \quad KF = v^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.$$

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring, and set for $i \in \mathbb{Z}$ and $n \geq 0$

$$[i] = \frac{v^i - v^{-i}}{v - v^{-1}}, \quad [n]! = [1][2] \cdots [n].$$

There are at least two well-known, interesting $\mathbb{Z}[v, v^{-1}]$-forms, $U_{\mathcal{A}}$ and $\overline{U}$, of U. $U_{\mathcal{A}}$ is defined to be the $\mathcal{A}$-subalgebra $U_{\mathcal{A}}$ of U generated by the elements K, K^{-1}, and the divided powers $E^{(n)} = E^n/[n]!$ and $F^{(n)} = F^n/[n]!$ for $n \geq 1$. $\overline{U}$ is defined to be the $\mathcal{A}$-subalgebra $\overline{U} \subset U_{\mathcal{A}}$ of U, which is generated by the elements K, K^{-1},

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$e = (v - v^{-1})E$, and $f = (v - v^{-1})F$. See [9,1] for the details of these two $\mathbb{Z}[v, v^{-1}]$-forms. (Both $U_{\mathcal{A}}$ and $\overline{U}$ are defined for finite dimensional simple Lie algebras, or more generally for Kac–Moody Lie algebras.) These algebras inherit Hopf $\mathcal{A}$-algebra structures from a Hopf algebra structure of U.

Let Θ denote the quasi-R-matrix of U

$$\Theta = \sum_{n\geq 0}(-1)^n v^{-\frac{1}{2}n(n-1)}(v - v^{-1})^n[n]!F^{(n)} \otimes E^{(n)}, \tag{1.1}$$

which is an element of a completion of $U \otimes U$. For the definition of the quasi-R-matrix, see [10,7]. It is well known that one can use Θ to obtain a left U-module isomorphism

$$b_{V,W} \colon V \otimes W \xrightarrow{\simeq} W \otimes V$$

for finite dimensional left U-modules V and W. For the category of finite dimensional left U-modules, the $b_{V,W}$ define a braided category structure, using which one can define the colored Jones polynomials of links.

One of the important properties of Θ is *integrality* in $U_{\mathcal{A}}$, i.e., each term in the infinite sum (1.1) is contained in $U_{\mathcal{A}}^{\otimes 2}$, and hence Θ is contained in a certain completion of $U_{\mathcal{A}}^{\otimes 2}$. One of the important consequences of this integrality in quantum topology is that the colored Jones polynomials of links take values in a Laurent polynomial ring.

We introduce another $\mathbb{Z}[v, v^{-1}]$-form $\mathcal{U}$ of U which is useful in the study of the quantum link invariants. This algebra may be regarded as a "mixed version" of $U_{\mathcal{A}}$ and $\overline{U}$.

Let $\mathcal{U}$ be the $\mathcal{A}$-subalgebra of U generated by the elements K, K^{-1}, e, and the $F^{(n)}$ for $n \geq 1$. Obviously, $\overline{U} \subset \mathcal{U} \subset U_{\mathcal{A}}$. We can rewrite (1.1) as follows:

$$\Theta = \sum_{n\geq 0}(-1)^n v^{-\frac{1}{2}n(n-1)} F^{(n)} \otimes e^n. \tag{1.2}$$

Observe that each term $(-1)^n v^{-\frac{1}{2}n(n-1)} F^{(n)} \otimes e^n$ is contained in $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{U}$. For each $n \geq 0$, let $\mathcal{U}_n^e = \mathcal{U}e^n\mathcal{U}$ be the two-sided ideal in $\mathcal{U}$ generated by the element e^n. Let $\check{\mathcal{U}}$ denote the completion

$$\check{\mathcal{U}} = \varprojlim_n \mathcal{U}/\mathcal{U}_n^e,$$

which will turn out to inherit from $\mathcal{U}$ a complete Hopf algebra structure over $\mathcal{A}$. The comultiplication $\check{\Delta} \colon \check{\mathcal{U}} \to \check{\mathcal{U}} \check{\otimes} \check{\mathcal{U}}$ takes values in the completed tensor product

$$\check{\mathcal{U}} \check{\otimes} \check{\mathcal{U}} = \varprojlim_{k,l} (\check{\mathcal{U}} \otimes_{\mathcal{A}} \check{\mathcal{U}})/(\overline{\mathcal{U}_k^e} \otimes_{\mathcal{A}} \check{\mathcal{U}} + \check{\mathcal{U}} \otimes_{\mathcal{A}} \overline{\mathcal{U}_l^e}),$$

where $\overline{\mathcal{U}_n^e}$ is the closure of $\mathcal{U}_n^e$ in $\check{\mathcal{U}}$. Then we can regard Θ as an element of $\check{\mathcal{U}} \check{\otimes} \check{\mathcal{U}}$. Unfortunately, the structure of $\check{\mathcal{U}}$ seems quite complicated. Therefore we also consider some other completions of $\mathcal{U}$, which have more controllable structures than $\check{\mathcal{U}}$.

For $i \in \mathbb{Z}$ and $n \geq 0$, set

$$\{i\} = v^i - v^{-i}, \quad \{n\}! = \{1\} \cdots \{n\} = (v - v^{-1})^n [n]!.$$

Let $\hat{\mathcal{A}}$ and $\dot{\mathcal{A}}$ denote the completions of $\mathcal{A}$ defined by

$$\hat{\mathcal{A}} = \varprojlim_n \mathcal{A}/(\{n\}!), \quad \dot{\mathcal{A}} = \varprojlim_n \mathcal{A}/((v - v^{-1})^n).$$

Since $\{n\}!$ is divisible by $(v-v^{-1})^n$, there is a natural homomorphism $\hat{\mathcal{A}} \to \dot{\mathcal{A}}$, which is injective (see Proposition 6.1). We regard $\hat{\mathcal{A}} \subset \dot{\mathcal{A}}$. For each $n \geq 0$, let $\mathcal{U}_n$ denote the two-sided ideal in $\mathcal{U}$ generated by the elements

$$\{H + m\}_i e^{n-i} \quad (m \in \mathbb{Z},\ 0 \leq i \leq n),$$

where

$$\{H + m\}_i = \prod_{k=0}^{i-1} (v^{m-i} K - v^{-m+i} K^{-1}).$$

It turns out that the two-sided ideal $\mathcal{U}_1$ of $\mathcal{U}$ is generated by the elements $v - v^{-1}$, $K - K^{-1}$, and e. We define the completions $\hat{\mathcal{U}}$ and $\dot{\mathcal{U}}$ of $\mathcal{U}$ by

$$\hat{\mathcal{U}} = \varprojlim_n \mathcal{U}/\mathcal{U}_n, \quad \dot{\mathcal{U}} = \varprojlim_n \mathcal{U}/(\mathcal{U}_1)^n,$$

which will be shown to have complete Hopf algebra structures over $\hat{\mathcal{A}}$ and $\dot{\mathcal{A}}$, respectively.

Let $U_h = U_h(sl_2)$ be the h-adic version of $U_v(sl_2)$, for the definition of which see Section 2. The algebra U_h contains $U_{\mathcal{A}}$, and hence $\overline{U}$ and $\mathcal{U}$. We will show that there is a sequence of natural homomorphisms

$$\check{\mathcal{U}} \to \hat{\mathcal{U}} \to \dot{\mathcal{U}} \to U_h,$$

where the two arrows $\hat{\mathcal{U}} \to \dot{\mathcal{U}} \to U_h$ are injective, and the arrow $\check{\mathcal{U}} \to \hat{\mathcal{U}}$ is conjectured to be injective. We set $\tilde{\mathcal{U}} = \operatorname{Im}(\check{\mathcal{U}} \to \hat{\mathcal{U}}) \subset \hat{\mathcal{U}}$. We have

$$\tilde{\mathcal{U}} \subset \hat{\mathcal{U}} \subset \dot{\mathcal{U}} \subset U_h.$$

We will determine the centers of $\mathcal{U}, \hat{\mathcal{U}}, \dot{\mathcal{U}}$, and $\tilde{\mathcal{U}}$ as follows. Set

$$C = (v - v^{-1})Fe + vK + v^{-1}K^{-1} \in \mathcal{U}.$$

Since C is central in U_h and in U (see e.g. [7, Section 2.7]), it follows that C is central in $\mathcal{U}$.

For $n \geq 0$, set

$$\sigma_n = \prod_{i=1}^{n} (C^2 - (v^i + v^{-i})^2).$$

Note that σ_n is a monic polynomial of degree n in C^2.

Theorem 1.1. *The center $Z(\mathcal{U})$ of $\mathcal{U}$ is freely generated by C as an $\mathcal{A}$-algebra, i.e., $Z(\mathcal{U}) = \mathcal{A}[C]$. The centers of $\tilde{\mathcal{U}}$, $\hat{\mathcal{U}}$, and $\dot{\mathcal{U}}$ are identified with completions of $\mathcal{A}[C]$ as follows:*

$$Z(\tilde{\mathcal{U}}) \simeq \varprojlim_n \mathcal{A}[C]/(\sigma_n), \quad Z(\hat{\mathcal{U}}) \simeq \varprojlim_n \hat{\mathcal{A}}[C]/(\sigma_n),$$

$$Z(\dot{\mathcal{U}}) \simeq \varprojlim_n \dot{\mathcal{A}}[C]/(\sigma_n) \simeq \varprojlim_n \dot{\mathcal{A}}[C]/((C^2 - [2]^2)^n).$$

Therefore $Z(\tilde{\mathcal{U}})$ (resp. $Z(\hat{\mathcal{U}})$, $Z(\dot{\mathcal{U}})$) consists of the elements which are uniquely expressed as infinite sums

$$z = \sum_{n \geq 0} z_n \sigma_n,$$

where $z_n \in \mathcal{A} + \mathcal{A}C$ (resp. $z_n \in \hat{\mathcal{A}} + \hat{\mathcal{A}}C$, $z_n \in \dot{\mathcal{A}} + \dot{\mathcal{A}}C$) for $n \geq 0$.

The proof of Theorem 1.1 is divided into Theorems 9.2, 9.9 and 9.13–9.15.

The results of the present paper are used in [5] where we prove integrality results of the sl_2 quantum invariants, announced in [2]. For a string knot L (i.e., a $(1, 1)$-tangle consisting only of one string) with 0 framing, let $J_L \in Z(U_h)$ denote the *universal sl_2 invariant* of L ([8,11]; see also [2,4]). This invariant has a universality property over the colored Jones polynomials of knots: The colored Jones polynomial of the knot obtained by closing the two ends of L, associated with a finite dimensional irreducible representation V of U_h, is obtained from J_L by taking its quantum trace in V. Using [4, Corollary 9.15], one can prove that J_L is contained in $Z(\mathcal{G}_0\tilde{\mathcal{U}}_q)$, where $\tilde{\mathcal{U}}_q$ is a $\mathbb{Z}[q, q^{-1}]$-form of $\tilde{\mathcal{U}}$ defined in Section 11, where $q = v^2$. Using Theorem 11.2, which is a slight modification of Theorem 1.1, we obtain the following result whose detailed proof can be found in [5].

Theorem 1.2. *Let L be a string knot with 0 framing. Then there are unique elements $a_n(L) \in \mathbb{Z}[q, q^{-1}]$ for $n \geq 0$ such that*

$$J_L = \sum_{n \geq 0} a_n(L)\sigma_n.$$

For consequences of Theorem 1.2, see [2]. In particular, using Theorem 1.2 and some other results in the present paper, we prove in [5] the existence of an invariant $I(M)$ of integral homology 3-spheres M with values in the ring

$$\varprojlim_n \mathbb{Z}[q,q^{-1}]/((1-q)(1-q^2)\cdots(1-q^n))$$

which specializes at each root ζ of unity to the Witten–Reshetikhin–Turaev invariants of M at ζ.

In a joint work with Le [6], we will generalize part of this paper to the quantized enveloping algebras of finite dimensional semisimple Lie algebras, and prove that the invariant $I(M)$ mentioned above generalizes to such Lie algebras.

The motivation of this work is, as explained above, from the study of quantum invariants. The rest of the paper, however, is purely algebraic. It is organized as follows. Section 2 recalls the definitions and the necessary properties of the quantized enveloping algebras $U_v(sl_2)$ and $U_h(sl_2)$, as well as the integral forms $U_{\mathcal{A}}$ and $\overline{U}$. We will describe in Section 3 the Hopf algebra structure of our integral form $\mathcal{U}$. In Section 4, we will introduce generalities on filtrations and completions of Hopf algebras. In Section 5, we will study the decreasing filtration $\{\mathcal{U}_n\}_n$ of $\mathcal{U}$, and the completion $\hat{\mathcal{U}}$. In Section 6, we will consider the $\mathcal{U}_1$-adic filtration of $\mathcal{U}$, study the completion $\dot{\mathcal{U}}$, and prove the injectivity of $\hat{\mathcal{U}} \to \dot{\mathcal{U}}$ and $\dot{\mathcal{U}} \to U_h$. In Section 7, we study the completion $\check{\mathcal{U}}$ and the algebra $\tilde{\mathcal{U}}$. Section 8 introduces $(\mathbb{Z}/2\mathbb{Z})$-gradings on the algebra $\mathcal{U}$ and the other algebras, which count "degrees in K" in the sense that the grading for $\mathcal{U}$ is as $\mathcal{U} = \mathcal{G}_0\mathcal{U} \oplus \mathcal{G}_1\mathcal{U}$, $\mathcal{G}_1\mathcal{U} = K\mathcal{G}_0\mathcal{U}$. We determine the centers of $\mathcal{U}$, $\hat{\mathcal{U}}$, $\tilde{\mathcal{U}}$, and $\dot{\mathcal{U}}$ in Sections 9 and 10 using the Harish–Chandra homomorphism. Section 11 gives $\mathbb{Z}[q,q^{-1}]$-forms of $\mathcal{U}$ and its completions.

2. The quantized enveloping algebras $U_v(sl_2)$ and $U_h(sl_2)$

In this section, we will recall the definitions and properties of the algebras $U_v(sl_2)$ and $U_h(sl_2)$, and fix notation.

2.1. Base rings and fields

Set $q = v^2$. We will also need the rational function field $\mathbb{Q}(q) \subset \mathbb{Q}(v)$ and the Laurent polynomial ring $\mathcal{A}_q = \mathbb{Z}[q,q^{-1}] \subset \mathcal{A}$. We have $\mathcal{A}_q = \mathcal{A} \cap \mathbb{Q}(q)$, and

$$\mathbb{Q}(v) = \mathbb{Q}(q) \oplus v\mathbb{Q}(q), \quad \mathcal{A} = \mathcal{A}_q \oplus v\mathcal{A}_q.$$

Let h be an indeterminate, and let $\mathbb{Q}[[h]]$ denote the formal power series ring. We will regard $\mathcal{A}$ and $\mathcal{A}_q$ as subrings of $\mathbb{Q}[[h]]$ by setting

$$v = \exp\frac{h}{2} \in \mathbb{Q}[[h]], \quad q = v^2 = \exp h \in \mathbb{Q}[[h]].$$

The quotient field $\mathbb{Q}((h))$ of $\mathbb{Q}[[h]]$ contains the fields $\mathbb{Q}(v)$ and $\mathbb{Q}(q)$ as subfields.

2.2. Algebra structures of U_h and U

The h-adic quantized universal enveloping algebra $U_h = U_h(sl_2)$ is defined to be the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements H, E, and F, subject to the relations

$$HE = E(H+2), \quad HF = F(H-2), \quad EF - FE = \frac{K-K^{-1}}{v-v^{-1}},$$

where we set

$$K = v^H = \exp\frac{1}{2}hH.$$

The algebras U_h and U are compatible in the sense that they can naturally be regarded as subalgebras of the $\mathbb{Q}((h))$-algebra $U_h \otimes_{\mathbb{Q}[[h]]} \mathbb{Q}((h))$.

Let U_h^0 (resp. U_h^+, U_h^-) denote the closure of the $\mathbb{Q}[[h]]$-subalgebra of U_h generated by H (resp. E, F), and let U^0 (resp., U^+, U^-) denote the $\mathbb{Q}(v)$-subalgebra of U generated by $K^{\pm 1}$ (resp. E, F). We have the triangular decompositions

$$U_h \simeq U_h^- \hat{\otimes}_h U_h^0 \hat{\otimes}_h U_h^+, \quad U \simeq U^- \otimes_{\mathbb{Q}(v)} U^0 \otimes_{\mathbb{Q}(v)} U^+,$$

where $\hat{\otimes}_h$ denotes the h-adically completed tensor product over $\mathbb{Q}[[h]]$.

The algebra U has a $\mathbb{Z}$-graded $\mathbb{Q}(v)$-algebra structure with the degree determined by

$$\deg K^{\pm 1} = 0, \quad \deg E = 1, \quad \deg F = -1.$$

For each $n \in \mathbb{Z}$, let $\Gamma_n U$ denote the nth graded part of U. We have

$$\begin{aligned}\Gamma_n U &= \{x \in U \mid K x K^{-1} = v^{2n} x\} \\ &= \mathrm{Span}_{\mathbb{Q}(v)}\{F^i K^j E^k \mid j \in \mathbb{Z}, i, k \geq 0, k - i = n\}.\end{aligned}$$

Similar $\mathbb{Z}$-grading on U_h is defined and let $\Gamma_n U_h$ denote the nth graded part of U_h. For a homogeneous element x in U or U_h, let $|x|$ denote the degree of x. For $n \in \mathbb{Z}$ and any homogeneous additive subgroup G of U (resp. U_h), we set

$$\Gamma_n G = \Gamma_n U \cap G \quad (\text{resp. } \Gamma_n U_h \cap G).$$

For each $j \in \mathbb{Z}$, the *shift automorphism*

$$\gamma_j : U_h^0 \to U_h^0$$

is the unique continuous automorphism of $\mathbb{Q}[[h]]$-algebra satisfying $\gamma_j(H) = H + j$. Similarly, the automorphism

$$\gamma_j : U^0 \to U^0$$

is defined by $\gamma_j(K) = v^j K = v^{H+j}$. We will freely use the fact that for any homogeneous element x in U_h (resp. U) and any element t of U_h^0 (resp. U^0), we have $tx = x\gamma_{2|x|}(t)$.

We use the following notation. For $a \in \mathbb{Z} + \mathbb{Z}H$ and $n \geq 0$,

$$\begin{aligned}&\{a\} = v^a - v^{-a},\\ &\{a\}_n = \{a\}\{a-1\}\cdots\{a-n+1\},\\ &\begin{bmatrix} a \\ n \end{bmatrix} = \frac{\{a\}_n}{\{n\}!} = \frac{[a][a-1]\cdots[a-n+1]}{[n]!}.\end{aligned}$$

If $a = iH + j$, $i, j \in \mathbb{Z}$, then $\{iH + j\} = v^j K^i - v^{-j} K^{-i}$.

2.3. Hopf algebra structures of U and U_h

The algebras U_h and U have compatible Hopf algebra structures with the comultiplication Δ, the counit ϵ and the antipode S defined by

$$\begin{aligned}&\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H,\\ &\Delta(K) = K \otimes K, \quad \epsilon(K) = 1, \quad S(K) = K^{-1},\\ &\Delta(E) = E \otimes 1 + K \otimes E, \quad \epsilon(E) = 0, \quad S(E) = -K^{-1}E,\\ &\Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \epsilon(F) = 0, \quad S(F) = -FK.\end{aligned}$$

Here U_h is an h-adic Hopf algebra over $\mathbb{Q}[[h]]$, and U is an (ordinary) Hopf algebra over $\mathbb{Q}(v)$.

2.4. The $\mathcal{A}$-forms $U_{\mathcal{A}}$ and $\overline{U}$ of U

In the introduction, we have defined the $\mathcal{A}$-subalgebras $U_{\mathcal{A}}$ and $\overline{U}$ of U, introduced in [9,1]. Note that they are also $\mathcal{A}$-subalgebras of U_h. For $* = 0, +, -$, set

$$U_{\mathcal{A}}^* = U^* \cap U_{\mathcal{A}} = U_h^* \cap U_{\mathcal{A}}, \quad \overline{U}^* = U^* \cap \overline{U} = U_h^* \cap \overline{U}.$$

It is known [9,1] that

$$U_{\mathcal{A}}^0 = \mathrm{Span}_{\mathcal{A}[K,K^{-1}]}\left\{ \begin{bmatrix} H \\ n \end{bmatrix} \middle| \, n \geq 0 \right\}, \tag{2.1}$$

$$U_{\mathcal{A}}^+ = \mathrm{Span}_{\mathcal{A}}\{E^{(n)} \mid n \geq 0\}, \quad U_{\mathcal{A}}^- = \mathrm{Span}_{\mathcal{A}}\{F^{(n)} \mid n \geq 0\}, \tag{2.2}$$

$$U^-_{\mathcal{A}} \otimes_{\mathcal{A}} U^0_{\mathcal{A}} \otimes_{\mathcal{A}} U^+_{\mathcal{A}} \xrightarrow{\simeq} U_{\mathcal{A}}, \quad x \otimes y \otimes z \mapsto xyz, \tag{2.3}$$

$$\overline{U}^0 = \mathcal{A}[K, K^{-1}], \quad \overline{U}^+ = \mathcal{A}[e], \quad \overline{U}^- = \mathcal{A}[f], \tag{2.4}$$

$$\overline{U}^- \otimes_{\mathcal{A}} \overline{U}^0 \otimes_{\mathcal{A}} \overline{U}^+ \xrightarrow{\simeq} \overline{U}, \quad x \otimes y \otimes z \mapsto xyz, \tag{2.5}$$

and that $U_{\mathcal{A}}$ and $\overline{U}$ inherit Hopf $\mathcal{A}$-algebra structures from that of U.

3. The $\mathcal{A}$-form $\mathcal{U}$ of U

3.1. The $\mathcal{A}$-form $\mathcal{U}$

Recall from the introduction that $\mathcal{U}$ is the $\mathcal{A}$-subalgebra of U generated by the elements K, K^{-1}, e, and the $F^{(n)}$ for $n \geq 1$. In other words, $\mathcal{U}$ is the smallest $\mathcal{A}$-subalgebra of U containing $\overline{U}^0 \cup \overline{U}^+ \cup U^-_{\mathcal{A}}$. As remarked in the introduction, we have $\overline{U} \subset \mathcal{U} \subset U_{\mathcal{A}}$. For $* = 0, +, -$, set

$$\mathcal{U}^* = U^* \cap \mathcal{U} = U^*_{\mathcal{A}} \cap \mathcal{U},$$

which are $\mathcal{A}$-subalgebras of $\mathcal{U}$.

Proposition 3.1. *We have*

$$\mathcal{U}^0 = \overline{U}^0, \quad \mathcal{U}^+ = \overline{U}^+, \quad \mathcal{U}^- = U^-_{\mathcal{A}}.$$

We have the triangular decomposition

$$\mathcal{U}^- \otimes_{\mathcal{A}} \mathcal{U}^0 \otimes_{\mathcal{A}} \mathcal{U}^+ \xrightarrow{\simeq} \mathcal{U}, \quad x \otimes y \otimes z \mapsto xyz.$$

Hence $\mathcal{U}$ is freely $\mathcal{A}$-spanned by the elements $F^{(m)} K^i e^n$ for $i \in \mathbb{Z}$, $m, n \geq 0$.

Proof. By induction, we see that for $m, n \geq 0$

$$e^m F^{(n)} = \sum_{p=0}^{\min(m,n)} \begin{bmatrix} m \\ p \end{bmatrix} F^{(n-p)} \{H - m - n + 2p\}_p e^{m-p}. \tag{3.1}$$

Hence $\overline{U}^+ U^-_{\mathcal{A}} \subset U^-_{\mathcal{A}} \overline{U}^0 \overline{U}^+$. We also have $\overline{U}^+ \overline{U}^0 = \overline{U}^0 \overline{U}^+$ and $\overline{U}^0 U^-_{\mathcal{A}} = U^-_{\mathcal{A}} \overline{U}^0$. Then we can easily verify that $U^-_{\mathcal{A}} \overline{U}^0 \overline{U}^+$ is an $\mathcal{A}$-subalgebra of $\mathcal{U}$. Since $U^-_{\mathcal{A}} \overline{U}^0 \overline{U}^+$ generates $\mathcal{U}$, we have $U^-_{\mathcal{A}} \overline{U}^0 \overline{U}^+ = \mathcal{U}$. Hence there is a surjective $\mathcal{A}$-module homomorphism $U^-_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{U}^0 \otimes_{\mathcal{A}} \overline{U}^+ \to \mathcal{U}$, $x \otimes y \otimes z \mapsto xyz$, which is injective in view of (2.3). Hence the statements follow. □

We do not need the following result in the rest of this paper, but we state it for completeness.

Proposition 3.2. *As an $\mathcal{A}$-algebra, $\mathcal{U}$ has a presentation with generators K, K^{-1}, e, and the $F^{(n)}$ for $n \geq 1$, and with the relations*

$$KK^{-1} = K^{-1}K = 1, \quad Ke = v^2 eK, \quad KF^{(n)} = v^{-2n} F^{(n)} K, \tag{3.2}$$

$$F^{(m)} F^{(n)} = \begin{bmatrix} m+n \\ m \end{bmatrix} F^{(m+n)} \quad (m, n \geq 1), \tag{3.3}$$

$$eF^{(n)} = F^{(n)} e + F^{(n-1)} (v^{-n+1} K - v^{n-1} K^{-1}) \quad (n \geq 1). \tag{3.4}$$

Proof. Relations (3.2)–(3.4) hold in $\mathcal{U}$. One can prove using only these relations that $\mathcal{U}$ is $\mathcal{A}$-spanned by the elements $F^{(i)} K^j e^k$ for $j \in \mathbb{Z}$, $i, k \geq 0$. It follows that the $\mathcal{A}$-algebra $\mathcal{U}$ is described by the generators and relations given above. □

3.2. Hopf algebra structure of $\mathcal{U}$

We will use the following formulas ($m \geq 0$):

$$\Delta(e) = e \otimes 1 + K \otimes e, \quad \epsilon(e) = 0, \quad S(e) = -K^{-1}e, \tag{3.5}$$

$$\Delta(e^m) = \sum_{i=0}^{m} v^{-i(m-i)} \begin{bmatrix} m \\ i \end{bmatrix} K^i e^{m-i} \otimes e^i, \tag{3.6}$$

$$\epsilon(e^m) = \delta_{m,0}, \quad S(e^m) = (-1)^m v^{m(m-1)} K^{-m} e^m, \tag{3.7}$$

$$\Delta(F^{(m)}) = \sum_{i=0}^{m} v^{i(m-i)} F^{(m-i)} \otimes F^{(i)} K^{-m+i}, \tag{3.8}$$

$$\epsilon(F^{(m)}) = \delta_{m,0}, \quad S(F^{(m)}) = (-1)^m v^{-m(m-1)} F^{(m)} K^m. \tag{3.9}$$

For the proofs of these formulas, see for example [7], where the formulas for e and e^m above can be easily derived from corresponding formulas for E and E^m, respectively.

Set $\mathcal{U}^{\geq 0} = \mathcal{U}^0 \mathcal{U}^+ = \overline{U}^0 \overline{U}^+$, which is an $\mathcal{A}$-subalgebra of $\mathcal{U}$.

Proposition 3.3. *The $\mathcal{A}$-algebra $\mathcal{U}$ is a Hopf $\mathcal{A}$-subalgebra of $U_{\mathcal{A}}$. The $\mathcal{A}$-subalgebras $\mathcal{U}^0$ and $\mathcal{U}^{\geq 0}$ of $\mathcal{U}$ are Hopf $\mathcal{A}$-subalgebras of $\mathcal{U}$.*

Proof. It is well known that $\mathcal{U}^0 = \overline{U}^0$ and $\mathcal{U}^{\geq 0} = \overline{U}^0 \overline{U}^+$ are Hopf $\mathcal{A}$-subalgebras of $\overline{U}$. Therefore the first statement implies the second. By (3.8) and (3.9),

$$\Delta(\mathcal{U}^-) \subset \mathcal{U}^- \otimes \mathcal{U}^- \mathcal{U}^0, \quad \epsilon(\mathcal{U}^-) \subset \mathcal{A}, \quad S(\mathcal{U}^-) \subset \mathcal{U}^- \mathcal{U}^0.$$

Using the triangular decomposition, we can easily check that

$$\Delta(\mathcal{U}) \subset \mathcal{U} \otimes \mathcal{U}, \quad \epsilon(\mathcal{U}) = \mathcal{A}, \quad S(\mathcal{U}) \subset \mathcal{U},$$

which implies the first statement. The second statement follows since $\mathcal{U}^0 = \overline{U}^0$ and $\mathcal{U}^{\geq 0} = \overline{U}^0 \overline{U}^+$ are Hopf $\mathcal{A}$-subalgebras of $\overline{U}$. □

As one can easily see, $\mathcal{U}^{\leq 0} = \mathcal{U}^- \mathcal{U}^0$ also is a Hopf $\mathcal{A}$-subalgebra of $\mathcal{U}$, but we will not need this fact.

4. Hopf algebra filtrations

In this subsection, we fix terminology on filtrations and linear topology for Hopf algebras.

Let R be a commutative ring with unit, and let

$$R = R_0 \supset R_1 \supset \cdots \supset R_n \supset \cdots \tag{4.1}$$

be a decreasing filtration of ideals in R. (Here we do *not* assume that $R_m R_n \subset R_{m+n}$.) The filtration $\{R_n\}_n$ defines a linear topology of R, and the completion $\hat{R} = \varprojlim_n R/R_n$ is a commutative R-algebra. If $R_1 = 0$, then $\{R_n\}_n$ is said to be *discrete*, and we identify $\hat{R}$ with R.

Let $A = (A, \Delta, \epsilon, S)$ be a Hopf R-algebra with comultiplication Δ, counit ϵ, and antipode S. Suppose for simplicity that A is a free R-module. A *Hopf algebra filtration* of A with respect to a decreasing filtration $\{R_n\}_n$ of two-sided ideals in R will mean a decreasing family of two-sided ideals in A

$$A = A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots \tag{4.2}$$

such that

$$R_n \subset A_n, \tag{4.3}$$

$$\Delta(A_n) \subset \sum_{i+j=n} A_i \otimes_R A_j, \tag{4.4}$$

$$\epsilon(A_n) \subset R_n, \tag{4.5}$$
$$S(A_n) \subset A_n, \tag{4.6}$$

for each $n \geq 0$. (We do *not* assume $A_m A_n \subset A_{m+n}, m, n \geq 0$.) If the filtration $\{R_n\}_n$ is discrete, then $\{A_n\}_n$ is called a Hopf algebra filtration over R. The completion $\hat{A} = \varprojlim_n A/A_n$ is a complete Hopf algebra over $\hat{R}$ with the structure morphisms

$$\hat{\Delta}\colon \hat{A} \to \hat{A}\hat{\otimes}\hat{A}, \quad \hat{\epsilon}\colon \hat{A} \to \hat{R}, \quad \hat{S}\colon \hat{A} \to \hat{A}$$

induced by those of A. Here $\hat{A}\hat{\otimes}\hat{A}$ is the completed tensor product of two copies of $\hat{A}$ defined by

$$\begin{aligned}\hat{A}\hat{\otimes}\hat{A} &= \varprojlim_{k,l}(\hat{A} \otimes_{\hat{R}} \hat{A})/(\bar{A}_k \otimes_{\hat{R}} \hat{A} + \hat{A} \otimes_{\hat{R}} \bar{A}_l) \\ &\simeq \varprojlim_{k,l}(A \otimes_R A)/(A_k \otimes_R A + A \otimes_R A_l),\end{aligned}$$

where $\bar{A}_k$ is the closure of A_k in $\hat{A}$.

Let A be a Hopf algebra over a commutative ring R. Let I be an ideal in R. A two-sided ideal J in A is called a *Hopf ideal with respect to I* if

$$I \subset J, \quad \Delta(J) \subset J \otimes_R A + A \otimes_R J, \quad S(J) \subset J, \quad \epsilon(J) \subset I.$$

In this case, the J-adic filtration $\{J^n\}_n$ of A is a Hopf algebra filtration with respect to $\{I^n\}_n$, and hence the completion $\varprojlim_n A/J^n$ is a complete Hopf algebra over $\hat{R} = \varprojlim_n R/I^n$. Note that a Hopf ideal with respect to (0) is just a Hopf ideal in the usual sense.

The filtration (4.1) of R will be called *fine* if

$$R_{m+n} \subset R_m R_n \quad (m, n \geq 0).$$

(Note the direction of the inclusion.) The Hopf algebra filtration (4.2) will be called *fine* if

$$A_{m+n} \subset A_m A_n \quad (m, n \geq 0). \tag{4.7}$$

Suppose that $\{R_n\}_n$ is a fine filtration of a commutative ring R with unit, and $\{A_n\}_n$ is a fine Hopf algebra filtration of a Hopf algebra A with respect to $\{R_n\}_n$. Then the two-sided ideal A_1 in A is a Hopf ideal with respect to R_1. Moreover, for each $n \geq 0$,

$$R_n \subset R_1^n, \quad A_n \subset A_1^n.$$

Hence id_R induces a continuous ring homomorphism

$$\varprojlim_n R/R_n \to \varprojlim_n R/R_1^n,$$

and id_A induces a continuous $(\varprojlim_n R/R_n)$-algebra homomorphism

$$\varprojlim_n A/A_n \to \varprojlim_n A/A_1^n.$$

5. The filtration $\{\mathcal{U}_n\}_n$ of $\mathcal{U}$ and the completion $\hat{\mathcal{U}}$

5.1. *The filtration $\{\mathcal{A}_n\}_n$ of $\mathcal{A}$ and the completion $\hat{\mathcal{A}}$*

For $n \geq 0$, set

$$\begin{aligned}&(q)_n = (1-q)(1-q^2)\cdots(1-q^n) = (-1)^n v^{\frac{1}{2}n(n+1)}\{n\}!,\\ &\mathcal{A}_n = (q)_n\mathcal{A} = \{n\}!\mathcal{A}, \quad (\mathcal{A}_q)_n = (q)_n\mathcal{A}_q = \mathcal{A}_n \cap \mathcal{A}_q.\end{aligned}$$

Since $\{m+n\}! = \{m\}!\{n\}! \begin{bmatrix} m+n \\ m \end{bmatrix} \in \mathcal{A}_m\mathcal{A}_n$ for $m, n \geq 0$, the filtrations $\{\mathcal{A}_n\}_n$ and $\{(\mathcal{A}_q)_n\}_n$ are fine. Set

$$\hat{\mathcal{A}} = \varprojlim_n \mathcal{A}/\mathcal{A}_n, \quad \hat{\mathcal{A}}_q = \varprojlim_n \mathcal{A}_q/(\mathcal{A}_q)_n.$$

Note that the definition of $\hat{\mathcal{A}}$ is the same as in the introduction. The $(\mathbb{Z}/2\mathbb{Z})$-grading $\mathcal{A} = \mathcal{A}_q \oplus v\mathcal{A}_q$ of $\mathcal{A}$ induces the $(\mathbb{Z}/2\mathbb{Z})$-grading $\hat{\mathcal{A}} = \hat{\mathcal{A}}_q \oplus v\hat{\mathcal{A}}_q$ of $\hat{\mathcal{A}}$. The ring $\hat{\mathcal{A}}_q$ is studied in [3], where it is denoted by $\mathbb{Z}[q]^{\mathbb{N}}$.

5.2. The filtration $\{\mathcal{U}^0_n\}_n$ of $\mathcal{U}^0$ and the completion $\hat{\mathcal{U}}^0$

For each $n \geq 0$, let $\mathcal{U}^0_n$ denote the two-sided ideal in $\mathcal{U}^0$ generated by the elements $\{H+m\}_n$ for $m \in \mathbb{Z}$. Since $\gamma_j(\{H+m\}_n) = \{H+m+j\}_n$ for each $j \in \mathbb{Z}$, the ideal $\mathcal{U}^0_n$ of $\mathcal{U}^0$ is preserved by γ_j, i.e.,

$$\gamma_j(\mathcal{U}^0_n) = \mathcal{U}^0_n. \tag{5.1}$$

Hence, for any homogeneous $\mathcal{A}$-submodule M of $\mathcal{U}$, we have $M\mathcal{U}^0_n = \mathcal{U}^0_n M$, which we will freely use in what follows. Another consequence of (5.1) is that $\gamma_j : \mathcal{U}^0 \to \mathcal{U}^0$ induces a continuous $\hat{\mathcal{A}}$-algebra automorphism

$$\gamma_j : \hat{\mathcal{U}}^0 \to \hat{\mathcal{U}}^0.$$

Since $\mathcal{U}^0 = \mathcal{A}[K, K^{-1}]$ is Noetherian, each ideal $\mathcal{U}^0_n$ is finitely generated. In fact, the following holds.

Proposition 5.1. *For each $n \geq 0$, the ideal $\mathcal{U}^0_n$ in $\mathcal{U}^0$ is generated by the elements*

$$\{n\}_i\{H\}_{n-i}, \quad i = 0, 1, \ldots, n.$$

In particular, for each $n \geq 0$ the ideal $\mathcal{U}^0_n$ contains the element $\{n\}!$, and hence we have $\{n\}!\mathcal{U}^0 \subset \mathcal{U}^0_n$.

Proof. Write $c_{n,i} = \{n\}_i\{H\}_{n-i}$, and set $(\mathcal{U}^0_n)' = (c_{n,0}, c_{n,1}, \ldots, c_{n,n})$. We will show that $\mathcal{U}^0_n = (\mathcal{U}^0_n)'$. One can prove by induction that

$$\{H+m\}_n = \sum_{i=0}^{n} v^{(n-i)m} K^{-i} \begin{bmatrix} m \\ i \end{bmatrix} c_{n,i} \quad (m \in \mathbb{Z},\ n \geq 0). \tag{5.2}$$

Hence $\mathcal{U}^0_n \subset (\mathcal{U}^0_n)'$. To prove the other inclusion $(\mathcal{U}^0_n)' \subset \mathcal{U}^0_n$, we will see that $c_{n,p} \in \mathcal{U}^0_n$ for $p = 0, \ldots, n$ by induction on p, the case $p = 0$ being trivial. If $0 < p \leq n$, then by (5.2) we have $c_{n,p} \in \{H+p\}_n + (c_{n,0}, \ldots, c_{n,p-1})$, which is contained in $\mathcal{U}^0_n$ by the induction hypothesis. $\square$

Proposition 5.2. *The family $\{\mathcal{U}^0_n\}_n$ is a fine Hopf algebra filtration of $\mathcal{U}^0$ with respect to $\{\mathcal{A}_n\}_n$.*

Proof. We will verify conditions (4.3)–(4.7) for $H_n = \mathcal{U}^0_n$ and $R_n = \mathcal{A}_n$. Proposition 5.1 implies (4.3). One can verify by induction that

$$\Delta(\{H+m\}_n) = \sum_{i+j=n} v^{-im} \begin{bmatrix} n \\ i \end{bmatrix} K^{-i}\{H+m\}_j \otimes K^j\{H\}_i,$$

which implies (4.4). Since $\epsilon(\{H+m\}_n) = \{m\}_n = \begin{bmatrix} m \\ n \end{bmatrix}\{n\}! \in \mathcal{A}_n$, we have (4.5). Since $S(\{H+m\}_n) = \{-H+m\}_n = (-1)^n\{H-m+n-1\}_n \in \mathcal{U}^0_n$, we have (4.6). Since $\{H+m\}_{n+n'} = \{H+m\}_n\{H+m-n\}_{n'}$ for $m \in \mathbb{Z}, n, n' \geq 0$, we have (4.7). $\square$

Let $\hat{\mathcal{U}}^0$ denote the completion

$$\hat{\mathcal{U}}^0 = \varprojlim_n \mathcal{U}^0/\mathcal{U}^0_n,$$

which is a complete Hopf $\hat{\mathcal{A}}$-algebra.

5.3. A double filtration of $\mathcal{U}^0$ cofinal with $\{\mathcal{U}^0_n\}_n$

For $k, l \geq 0$, set

$$\mathcal{U}^0_{k,l} = (\{k\}!, \{H\}_l) \subset \mathcal{U}^0.$$

The family $\{\mathcal{U}^0_{k,l}\}_{k,l\geq 0}$ forms a decreasing double filtration of $\mathcal{U}$, i.e., $\mathcal{U}^0_{k',l'} \subset \mathcal{U}^0_{k,l}$ if $0 \leq k \leq k'$ and $0 \leq l \leq l'$.

Proposition 5.3. *The double filtration $\{\mathcal{U}^0_{k,l}\}_{k,l\geq 0}$ is cofinal with $\{\mathcal{U}^0_n\}_n$.*

Proof. By Proposition 5.1, we have $\mathcal{U}^0_{n,n} = (\{n\}!, \{H\}_n) \subset \mathcal{U}^0_n$ for $n \geq 0$, and $\mathcal{U}^0_{2n-1} \subset (\{n\}!, \{H\}_n) = \mathcal{U}^0_{n,n}$ for $n \geq 1$. Therefore $\{\mathcal{U}^0_n\}_n$ is cofinal with $\{\mathcal{U}^0_{n,n}\}_n$, and hence with $\{\mathcal{U}^0_{k,l}\}_{k,l\geq 0}$. □

For $l \geq 0$, set

$$\{H\}'_l = (K^2 - 1)(K^2 - q)\cdots(K^2 - q^{l-1}) = v^{l(l-1)/2} K^l \{H\}_l \in \mathcal{A}_q[K^2, K^{-2}].$$

Proposition 5.4. *There are natural isomorphisms*

$$\hat{\mathcal{U}}^0 \simeq \varprojlim_l (\hat{\mathcal{A}}[K]/(\{H\}'_l)) \simeq \varprojlim_l (\hat{\mathcal{A}}[K, K^{-1}]/(\{H\}_l)). \tag{5.3}$$

Proof. By Proposition 5.3,

$$\hat{\mathcal{U}}^0 \simeq \varprojlim_{k,l} \mathcal{U}^0/\mathcal{U}^0_{k,l} \simeq \varprojlim_l \left(\varprojlim_k \mathcal{A}[K, K^{-1}]/(\{k\}!, \{H\}'_l) \right).$$

Note that $\{H\}'_l \in \mathcal{A}_q[K^2] \subset \mathcal{A}[K]$ is a monic polynomial of degree $2l$ in K with coefficients in $\mathcal{A}$, and the 0th-degree coefficient of $\{H\}'_l$ is a unit in $\mathcal{A}$. Hence

$$\begin{aligned}\varprojlim_k \mathcal{A}[K, K^{-1}]/(\{k\}!, \{H\}'_l) &\simeq \varprojlim_k \mathcal{A}[K]/(\{k\}!, \{H\}'_l)\\ &\simeq \varprojlim_k ((\mathcal{A}/(\{k\}!))[K]/(\{H\}'_l)) \simeq \hat{\mathcal{A}}[K]/(\{H\}'_l).\end{aligned}$$

Therefore $\hat{\mathcal{U}}^0 \simeq \varprojlim_l \hat{\mathcal{A}}[K]/(\{H\}'_l)$. The second isomorphism in (5.3) is obvious from the above argument. □

Proposition 5.4 implies the following.

Corollary 5.5. *Each element $t \in \hat{\mathcal{U}}^0$ is uniquely expressed as an infinite sum $t = \sum_{n\geq 0} t_n \{H\}'_l$, where $t_n \in \hat{\mathcal{A}} + \hat{\mathcal{A}}K$ for $n \geq 0$.*

5.4. The filtration $\{\mathcal{U}^{\geq 0}_n\}_n$ of $\mathcal{U}^{\geq 0}$ and the completion $\hat{\mathcal{U}}^{\geq 0}$

For $n \geq 0$, let $\mathcal{U}^+_n$ denote the principal ideal in $\mathcal{U}^+_n = \mathcal{A}[e]$ generated by the element e^n, and set

$$\mathcal{U}^{\geq 0}_n = \sum_{i+j=n} \mathcal{U}^0_i \mathcal{U}^+_j = \sum_{i+j=n} \mathcal{U}^+_j \mathcal{U}^0_i \subset \mathcal{U}^{\geq 0}.$$

Using $\mathcal{U}^{\geq 0}\mathcal{U}^0_i = \mathcal{U}^0_i \mathcal{U}^{\geq 0}$, we easily see that $\mathcal{U}^{\geq 0}$ is a two-sided ideal in $\mathcal{U}^{\geq 0}$. Obviously, the family $\{\mathcal{U}^{\geq 0}_n\}_n$ is a decreasing filtration.

Proposition 5.6. *The family $\{\mathcal{U}^{\geq 0}_n\}_n$ is a fine Hopf algebra filtration of $\mathcal{U}^{\geq 0}$ with respect to $\{\mathcal{A}_n\}_n$.*

Proof. We will verify conditions (4.3)–(4.7) for $H_n = \mathcal{U}^{\geq 0}_n$ and $R_n = \mathcal{A}_n$ using Proposition 5.2. We clearly have (4.3). The inclusion (4.4) follows from Proposition 5.2 and

$$\Delta(\mathcal{U}^+_n) \subset \sum_{i+j=n} \mathcal{U}^0 \mathcal{U}^+_i \otimes_{\mathcal{A}} \mathcal{U}^+_j,$$

which follows from (3.6). Condition (4.5) follows from Proposition 5.2 and the fact that $\epsilon(\mathcal{U}_n^+) = 0$ for $n > 0$. We show (4.6) as follows:

$$S(\mathcal{U}_n^{\geq 0}) = \sum_{i+j=n} S(\mathcal{U}_i^0\mathcal{U}_j^+) \subset \sum_{i+j=n} \mathcal{U}^0\mathcal{U}_j^+\mathcal{U}_i^0 \subset \sum_{i+j=n} \mathcal{U}_i^0\mathcal{U}_j^+ = \mathcal{U}_n^{\geq 0}.$$

Here we used $S(\mathcal{U}_n^+) \subset \mathcal{U}^0\mathcal{U}_n^+$, which follows from (3.7). We see (4.7) as follows. Suppose $m+n = i+j$, $i, j \geq 0$. If $n \leq j$, then $\mathcal{U}_i^0\mathcal{U}_j^+ = \mathcal{U}_i^0\mathcal{U}_{j-n}^+\mathcal{U}_n^+ \subset \mathcal{U}_m^{\geq 0}\mathcal{U}_n^{\geq 0}$, and otherwise $\mathcal{U}_i^0\mathcal{U}_j^+ = \mathcal{U}_m^0\mathcal{U}_{i-m}^0\mathcal{U}_j^+ \subset \mathcal{U}_m^{\geq 0}\mathcal{U}_n^{\geq 0}$. Hence (4.7) follows. □

Let $\hat{\mathcal{U}}^{\geq 0}$ denote the completion

$$\hat{\mathcal{U}}^{\geq 0} = \varprojlim_n \mathcal{U}^{\geq 0}/\mathcal{U}_n^{\geq 0},$$

which is a complete Hopf $\hat{\mathcal{A}}$-algebra.

It is obvious that the filtration $\{\mathcal{U}_n^{\geq 0}\}_n$ is cofinal with the double filtration $\{\mathcal{U}_k^0\mathcal{U}^+ + \mathcal{U}^0\mathcal{U}_l^+\}_{k,l\geq 0}$. We have

$$\mathcal{U}^{\geq 0}/(\mathcal{U}_k^0\mathcal{U}^+ + \mathcal{U}^0\mathcal{U}_l^+) \simeq (\mathcal{U}^0 \otimes_{\mathcal{A}} \mathcal{U}^+)/(\mathcal{U}_k^0\mathcal{U}^+ + \mathcal{U}^0\mathcal{U}_l^+) \simeq (\mathcal{U}^0/\mathcal{U}_k^0) \otimes_{\mathcal{A}} (\mathcal{U}^+/\mathcal{U}_l^+).$$

Therefore

$$\hat{\mathcal{U}}^{\geq 0} \simeq \varprojlim_{k,l} \mathcal{U}^{\geq 0}/(\mathcal{U}_k^0\mathcal{U}^+ + \mathcal{U}^0\mathcal{U}_l^+) \simeq \varprojlim_{k,l} (\mathcal{U}^0/\mathcal{U}_k^0) \otimes_{\mathcal{A}} (\mathcal{U}^+/\mathcal{U}_l^+)$$

The right hand side is isomorphic to $\varprojlim_l \hat{\mathcal{U}}^0 \otimes_{\mathcal{A}} (\mathcal{U}^+/\mathcal{U}_l^+)$ since $\mathcal{U}^+/\mathcal{U}_l^+$ is a finitely generated free $\mathcal{A}$-module. Hence there is an isomorphism of complete $\hat{\mathcal{A}}$-modules

$$\varprojlim_k \hat{\mathcal{U}}^0 \otimes_{\mathcal{A}} (\mathcal{U}^+/\mathcal{U}_k^+) \xrightarrow{\simeq} \hat{\mathcal{U}}^{\geq 0}, \tag{5.4}$$

induced by $\mathcal{U}^0 \otimes_{\mathcal{A}} \mathcal{U}^+ \xrightarrow{\simeq} \mathcal{U}^{\geq 0}$, $x \otimes y \mapsto xy$.

Proposition 5.7. *Each element $a \in \hat{\mathcal{U}}^{\geq 0}$ is uniquely expressed as an infinite sum $a = \sum_{n\geq 0} t_n e^n$, where $t_n \in \hat{\mathcal{U}}^0$ for $n \geq 0$.*

Proof. The result follows using (5.4) since each element of $\varprojlim_k \hat{\mathcal{U}}^0 \otimes_{\mathcal{A}} (\mathcal{U}^+/\mathcal{U}_k^+)$ is uniquely expressed as $a = \sum_{n\geq 0} t_n \otimes e^n$ with $t_n \in \hat{\mathcal{U}}^0$ for $n \geq 0$. □

5.5. *The filtration $\{\mathcal{U}_n\}_n$ of $\mathcal{U}$ and the completion $\hat{\mathcal{U}}$*

We have defined in the introduction the two-sided ideal $\mathcal{U}_n$ in $\mathcal{U}$ for $n \geq 0$. We have

$$\mathcal{U}_n = \mathcal{U}\mathcal{U}_n^{\geq 0}\mathcal{U} = \mathcal{U}^-\mathcal{U}_n^{\geq 0}\mathcal{U}^-.$$

Here the second identity follows from $\mathcal{U} = \mathcal{U}^-\mathcal{U}^{\geq 0} = \mathcal{U}^{\geq 0}\mathcal{U}^-$. The family $\{\mathcal{U}_n\}_n$ is a decreasing filtration of two-sided ideals in $\mathcal{U}$. The following follows immediately from Propositions 3.3 and 5.6.

Proposition 5.8. *The filtration $\{\mathcal{U}_n\}_n$ is a fine Hopf algebra filtration of $\mathcal{U}$ with respect to $\{\mathcal{A}_n\}_n$.*

The completion $\hat{\mathcal{U}} = \varprojlim_n \mathcal{U}/\mathcal{U}_n$, already defined in the introduction, is a complete Hopf $\hat{\mathcal{A}}$-algebra.

5.6. *The filtration $\{\mathcal{U}'_n\}_n$*

We will regard $\mathcal{U}$ as a free right $\mathcal{U}^{\geq 0}$-module freely generated by the elements $F^{(i)}$ for $i \geq 0$, the right action being multiplication from the right. For each $n \geq 0$, set

$$\mathcal{U}'_n = \mathcal{U}\mathcal{U}_n^{\geq 0} = \mathcal{U}^-\mathcal{U}_n^{\geq 0},$$

which is the left ideal in $\mathcal{U}$ generated by $\mathcal{U}_n^{\geq 0}$, and is also a right $\mathcal{U}^{\geq 0}$-submodule of $\mathcal{U}$. The family $\{\mathcal{U}'_n\}_n$ is a decreasing filtration of left ideals in $\mathcal{U}$.

Proposition 5.9. *The filtration $\{\mathcal{U}'_n\}_n$ is cofinal with $\{\mathcal{U}_n\}_n$. In fact,*

$$\mathcal{U}_{2n-1} \subset \mathcal{U}'_n \subset \mathcal{U}_n \quad (n \geq 1). \tag{5.5}$$

Proof. Obviously, $\mathcal{U}'_n \subset \mathcal{U}_n$ for $n \geq 0$. By (3.1),

$$\mathcal{U}^+_m \mathcal{U}^- \subset \mathcal{U}^- \mathcal{U}^{\geq 0}_m$$

for all $m \geq 0$. Hence for $i, j \geq 0$

$$\mathcal{U}^- \mathcal{U}^0_i \mathcal{U}^+_j \mathcal{U}^- \subset \mathcal{U}^- \mathcal{U}^0_i \mathcal{U}^- \mathcal{U}^{\geq 0}_j = \mathcal{U}^- \mathcal{U}^0_i \mathcal{U}^{\geq 0}_j \subset \mathcal{U}^- \mathcal{U}^{\geq 0}_{\max(i,j)} = \mathcal{U}'_{\max(i,j)}.$$

Therefore

$$\mathcal{U}_{2n-1} = \sum_{i+j=2n-1} \mathcal{U}^- \mathcal{U}^0_i \mathcal{U}^+_j \mathcal{U}^- \subset \sum_{i+j=2n-1} \mathcal{U}'_{\max(i,j)} \subset \mathcal{U}'_n. \quad \square$$

By Proposition 5.9, there is a natural isomorphism

$$\hat{\mathcal{U}} \simeq \varprojlim_n \mathcal{U}/\mathcal{U}'_n = \varprojlim_n \mathcal{U}^- \mathcal{U}^{\geq 0}/\mathcal{U}^- \mathcal{U}^{\geq 0}_n.$$

By Proposition 3.1, we have

$$\hat{\mathcal{U}} \simeq \varprojlim_n \mathcal{U}^- \mathcal{U}^{\geq 0}/\mathcal{U}^- \mathcal{U}^{\geq 0}_n \simeq \varprojlim_n (\mathcal{U}^- \otimes_{\mathcal{A}} \mathcal{U}^{\geq 0})/(\mathcal{U}^- \otimes_{\mathcal{A}} \mathcal{U}^{\geq 0}_n) \simeq \varprojlim_n \mathcal{U}^- \otimes_{\mathcal{A}} (\mathcal{U}^{\geq 0}/\mathcal{U}^{\geq 0}_n).$$

Since $\mathcal{U}^-$ is freely $\mathcal{A}$-spanned by the elements $F^{(l)}$ for $l \geq 0$, the completion $\varprojlim_n \mathcal{U}^- \otimes (\mathcal{U}^{\geq 0}/\mathcal{U}^{\geq 0}_n)$ is the topologically free right $\hat{\mathcal{U}}^{\geq 0}$-module generated by the $F^{(l)} \otimes 1$ for $l \geq 0$. Hence we have the following.

Proposition 5.10. *The algebra $\hat{\mathcal{U}}$ is a topologically free right $\hat{\mathcal{U}}^{\geq 0}$-module which is topologically freely generated by the elements $F^{(n)}$ for $n \geq 0$. Consequently, each element $a \in \hat{\mathcal{U}}$ is uniquely expressed as an infinite sum $a = \sum_{n \geq 0} F^{(n)} a_n$, where the sequence $a_n \in \hat{\mathcal{U}}^{\geq 0}$ for $n \geq 0$ converges to 0 in $\hat{\mathcal{U}}^{\geq 0}$.*

Corollary 5.11. *Each element $a \in \hat{\mathcal{U}}$ is uniquely expressed as an infinite sum $a = \sum_{m,n \geq 0} F^{(n)} t_{m,n} e^m$, where the elements $t_{m,n} \in \hat{\mathcal{U}}^0$ $(m, n \geq 0)$ are such that for each $m \geq 0$ the sequence $(t_{m,n})_{n \geq 0}$ converges to 0 in $\hat{\mathcal{U}}^0$ as $n \to \infty$.*

Proof. We express a as in Proposition 5.10. By Proposition 5.7, we can express each a_n as $a_n = \sum_{m \geq 0} t_{m,n} e^m$, where $t_{m,n} \in \hat{\mathcal{U}}$ for each $m \geq 0$. The statement follows since a_n converges to 0 in $\hat{\mathcal{U}}^{\geq 0}$ as $n \to \infty$ if and only if for any $m \geq 0$ we have $t_{m,n} \to 0$ in $\hat{\mathcal{U}}^0$ as $n \to \infty$. $\square$

6. The $\mathcal{U}_1$-adic topology for $\mathcal{U}$ and the completion $\dot{\mathcal{U}}$

6.1. The $\mathcal{A}_1$-adic completion $\hat{\mathcal{A}}$ of $\mathcal{A}$

Note that

$$\mathcal{A}_1 = (v - v^{-1}) = (v^2 - 1) \subset \mathcal{A}, \quad (\mathcal{A}_q)_1 = (q - 1) \subset \mathcal{A}_q.$$

The completion $\dot{\mathcal{A}} = \varprojlim_n \mathcal{A}/\mathcal{A}^n_1$ defined in the introduction is $(\mathbb{Z}/2\mathbb{Z})$-graded: $\dot{\mathcal{A}} \simeq \dot{\mathcal{A}}_q \oplus v\dot{\mathcal{A}}_q$, where we set

$$\dot{\mathcal{A}}_q = \varprojlim_n \mathcal{A}_q/(\mathcal{A}_q)^n_1 = \varprojlim_n \mathbb{Z}[q, q^{-1}]/(q-1)^n = \mathbb{Z}[[q-1]].$$

As we mentioned in the introduction, $\mathrm{id}_{\mathcal{A}}$ induces an $\mathcal{A}$-algebra homomorphism $i_{\hat{\mathcal{A}}}\colon \hat{\mathcal{A}} \to \dot{\mathcal{A}}$. Since $v - v^{-1} \in h\mathbb{Q}[[h]]$, $\mathrm{id}_{\mathcal{A}}$ also induces an $\mathcal{A}$-algebra homomorphism $i_{\dot{\mathcal{A}}}\colon \dot{\mathcal{A}} \to \mathbb{Q}[[h]]$.

Proposition 6.1. *$i_{\hat{\mathcal{A}}}$ and $i_{\dot{\mathcal{A}}}$ are injective.*

Proof. $i_{\hat{\mathcal{A}}}$ is the direct sum of two homomorphisms

$$i_{\hat{\mathcal{A}}}|_{\hat{\mathcal{A}}_q} : \hat{\mathcal{A}}_q \to \dot{\mathcal{A}}_q, \quad i_{\hat{\mathcal{A}}}|_{v\hat{\mathcal{A}}_q} : v\hat{\mathcal{A}}_q \to v\dot{\mathcal{A}}_q.$$

The latter is injective if (and only if) the former is injective. The injectivity of the former is a special case of [3, Theorem 4.1], and is also obtained by P. Vogel (private communication). Hence $i_{\hat{\mathcal{A}}}$ is injective.

The homomorphism $i_{\hat{\mathcal{A}}}$ factors naturally as $\dot{\mathcal{A}} \to \mathbb{Z}[[v-1]] \to \mathbb{Q}[[h]]$. Here $\dot{\mathcal{A}} = \varprojlim_n \mathbb{Z}[v, v^{-1}]/(v^2-1)^n \to \mathbb{Z}[[v-1]]$ is injective by [3, Corollary 4.1], and the inclusion $\mathbb{Z}[[v-1]] \subset \mathbb{Q}[[h]]$ is standard. □

In what follows, we regard $\hat{\mathcal{A}} \subset \dot{\mathcal{A}} \subset \mathbb{Q}[[h]]$.

6.2. The $\mathcal{U}_1$-adic completion $\dot{\mathcal{U}}$ of $\mathcal{U}$

In view of Section 4, Propositions 5.2, 5.6 and 5.8, we have the following.

Proposition 6.2. *For $* = 0, \geq 0$, (none), $\mathcal{U}_1^*$ is a Hopf ideal in $\mathcal{U}^*$ with respect to $\mathcal{A}_1$. The $\mathcal{U}_1^*$-adic filtration $\{(\mathcal{U}_1^*)^n\}_n$ is a fine Hopf algebra filtration with respect to the $\mathcal{A}_1$-adic filtration $\{(\mathcal{A}_1)^n\}_n$ of $\mathcal{A}$.*

Set

$$\dot{\mathcal{U}}^0 = \varprojlim_n \mathcal{U}^0/(\mathcal{U}_1^0)^n, \quad \dot{\mathcal{U}}^{\geq 0} = \varprojlim_n \mathcal{U}^{\geq 0}/(\mathcal{U}_1^{\geq 0})^n, \quad \dot{\mathcal{U}} = \varprojlim_n \mathcal{U}/(\mathcal{U}_1)^n,$$

which are complete Hopf $\hat{\mathcal{A}}$-algebras. In view of Section 4, the identity maps of $\mathcal{U}^0, \mathcal{U}^{\geq 0}$, and $\mathcal{U}$ induce $\hat{\mathcal{A}}$-algebra homomorphisms

$$i_{\hat{\mathcal{U}}^0} : \hat{\mathcal{U}}^0 \to \dot{\mathcal{U}}^0, \quad i_{\hat{\mathcal{U}}^{\geq 0}} : \hat{\mathcal{U}}^{\geq 0} \to \dot{\mathcal{U}}^{\geq 0}, \quad i_{\hat{\mathcal{U}}} : \hat{\mathcal{U}} \to \dot{\mathcal{U}},$$

respectively.

Proposition 6.3. *For $n \geq 0$,*

$$(\mathcal{U}_1^0)^n = \sum_{i=0}^{n} (\{1\}^i \{H\}^{n-i}), \tag{6.1}$$

$$(\mathcal{U}_1^{\geq 0})^n = \sum_{i+j=n} (\mathcal{U}_1^0)^i \mathcal{U}_j^+, \tag{6.2}$$

$$(\mathcal{U}_1)^n = \mathcal{U}^- (\mathcal{U}_1^{\geq 0})^n, \tag{6.3}$$

$$(\mathcal{U}_1^0)^n = (\mathcal{U}_1)^n \cap \mathcal{U}^0, \quad (\mathcal{U}_1^{\geq 0})^n = (\mathcal{U}_1)^n \cap \mathcal{U}^{\geq 0}. \tag{6.4}$$

Proof. In view of Proposition 5.1, we have $\mathcal{U}_1^0 = (\{1\}, \{H\})$ and hence (6.1). Since $\{H\}e^n = e^n\{H+2n\} = e^n(v^{2n}\{H\} + \{2n\}K^{-1})$, it follows that $\mathcal{U}_1^0 e^n = e^n \mathcal{U}^0$, and hence $\mathcal{U}_1^0 \mathcal{U}_n^+ = \mathcal{U}_n^+ \mathcal{U}_1^0$. Using this identity, we have (6.2). By (5.5), $\mathcal{U}_1 = \mathcal{U}_1' = \mathcal{U}^- \mathcal{U}_1^{\geq 0}$. We also have $\mathcal{U}_1 = \mathcal{U}_1^{\geq 0} \mathcal{U}^-$, and hence $\mathcal{U}_1^{\geq 0} \mathcal{U}^- = \mathcal{U}^- \mathcal{U}_1^{\geq 0}$. Using this identity, we can easily verify (6.3) using induction. (6.4) follows from (6.2) and (6.3) and the triangular decomposition. □

In view of (6.4), the natural homomorphisms $\dot{\mathcal{U}}^0 \to \dot{\mathcal{U}}^{\geq 0} \to \dot{\mathcal{U}}$ are injective. In what follows, we will regard $\dot{\mathcal{U}}^0 \subset \dot{\mathcal{U}}^{\geq 0} \subset \dot{\mathcal{U}}$.

The following follows from $\mathcal{U}_1^0 = (v^2-1, K^2-1) \subset \mathcal{U}^0$.

Proposition 6.4. *We have*

$$\dot{\mathcal{U}}^0 \simeq \varprojlim_{k,l} \mathcal{U}^0/((v^2-1)^k, (K^2-1)^l) \simeq \mathbb{Z}[[v^2-1, K^2-1]] \otimes_{\mathbb{Z}[v^2,K^2]} \mathbb{Z}[v, K].$$

Hence $\dot{\mathcal{U}}^0$ consists of the elements uniquely expressed as power series

$$\sum_{i,j \geq 0} t_{i,j} (v^2-1)^i (K^2-1)^j,$$

where $t_{i,j} \in \mathbb{Z} \oplus \mathbb{Z}v \oplus \mathbb{Z}K \oplus \mathbb{Z}vK$ for $i, j \geq 0$.

Using arguments similar to those in Section 5.4, we obtain the following.

Proposition 6.5. *Each element of $\dot{\mathcal{U}}^{\geq 0}$ is uniquely expressed as an infinite sum $a = \sum_{n\geq 0} t_n e^n$, where $t_n \in \dot{\mathcal{U}}^0$ for $n \geq 0$.*

By an argument similar to Section 5.6, the completion $\dot{\mathcal{U}}$ is a topologically free right $\dot{\mathcal{U}}^{\geq 0}$-module generated by the elements $F^{(l)}$ for $l \geq 0$. Hence we have the following.

Proposition 6.6. *Each element $a \in \dot{\mathcal{U}}$ is uniquely expressed as an infinite sum $a = \sum_{m,n\geq 0} F^{(n)} t_{m,n} e^m$, where the elements $t_{m,n} \in \dot{\mathcal{U}}^0$ $(m, n \geq 0)$ are such that for each $m \geq 0$ the $t_{m,n}$ converges to 0 in $\dot{\mathcal{U}}^0$ as $n \to \infty$.*

6.3. Embedding of $\mathcal{U}$ into $\hat{\mathcal{U}}$ and $\dot{\mathcal{U}}$

Proposition 6.7. *We have $\bigcap_{n\geq 0} \mathcal{U}_n = \bigcap_{n\geq 0} (\mathcal{U}_1)^n = 0$. Consequently, the natural homomorphisms $\mathcal{U} \to \hat{\mathcal{U}}$, $\mathcal{U} \to \dot{\mathcal{U}}$ are injective.*

Proof. Since $v^2 - 1$, $K^2 - 1$, $e \in hU_h$, we have

$$\mathcal{U}_1 \subset hU_h. \tag{6.5}$$

Using (6.5) and the well-known fact that $\bigcap_{n\geq 0} (hU_h)^n = 0$, we conclude that $\bigcap_{n\geq 0} (\mathcal{U}_1)^n = 0$, and hence the statement follows. □

In what follows, we will regard $\mathcal{U} \subset \hat{\mathcal{U}}$ and $\mathcal{U} \subset \dot{\mathcal{U}}$ via the injective homomorphisms in Proposition 6.7.

6.4. Embedding of $\dot{\mathcal{U}}$ into U_h

By (6.5), there is an $\dot{\mathcal{A}}$-algebra homomorphism

$$i_{\dot{\mathcal{U}}} : \dot{\mathcal{U}} \to U_h$$

induced by $\mathcal{U} \subset U_h$. We have the following.

Proposition 6.8. *The homomorphism $i_{\dot{\mathcal{U}}}$ is injective.*

Proof. The injectivity of $i_{\dot{\mathcal{U}}}|_{\dot{\mathcal{U}}^0} : \dot{\mathcal{U}}^0 \to U_h^0$ follows, since we can factor it as a composition of natural injective homomorphisms as follows:

$$\begin{aligned} \dot{\mathcal{U}}^0 &\simeq \varprojlim_n \dot{\mathcal{A}}[K, K^{-1}]/(K^2 - 1)^n \xrightarrow{i_1} \varprojlim_n \mathbb{Z}[[v - 1]][K, K^{-1}]/(K^2 - 1)^n \\ &\xrightarrow{i_2} \mathbb{Z}[[v - 1, K - 1]] \subset \mathbb{Q}[[v - 1, K - 1]] \simeq \mathbb{Q}[[h, hH]] \xrightarrow{i_3} \mathbb{Q}[H][[h]] = U_h^0. \end{aligned}$$

Here, the map i_1 is injective since it is induced by $\dot{\mathcal{A}} \subset \mathbb{Z}[[v - 1]]$ (see the proof of Proposition 6.1). Moreover, the map i_2 is injective in view of [3, Corollary 4.1].

The injectivity of $i_{\dot{\mathcal{U}}}$ follows from the injectivity of $i_{\dot{\mathcal{U}}}|_{\dot{\mathcal{U}}^0}$ and Proposition 6.6. □

In what follows, we regard $\dot{\mathcal{U}}$ as an $\dot{\mathcal{A}}$-subalgebra of U_h via the injective homomorphism $i_{\dot{\mathcal{U}}}$.

6.5. Embedding of $\hat{\mathcal{U}}$ into $\dot{\mathcal{U}}$

Proposition 6.9. *For $* = 0, \geq 0$, (none), the homomorphism $i_{\hat{\mathcal{U}}^*}$ is injective.*

Proof. First consider the case $* = 0$. Let $t \in \hat{\mathcal{U}}^0$ and suppose that $i_{\hat{\mathcal{U}}^0}(t) = 0$. Express t as in Corollary 5.5. For $l \geq 1$, we can expand $\{H\}'_l$ in $K^2 - 1$ as

$$\{H\}'_l = \prod_{i=0}^{l-1} ((K^2 - 1) + (1 - v^{2i})) = \sum_{j=1}^{l} b_{l,j} (v^2 - 1)^{l-j} (K^2 - 1)^j$$

where $b_{l,j} \in \mathcal{A}$ for $j = 1, \ldots, l$, and $b_{l,l} = 1$. We have

$$i_{\hat{\mathcal{U}}^0}(t) = t_0 + \sum_{l=1}^{\infty} t_l \sum_{j=1}^{l} b_{l,j}(v^2-1)^{l-j}(K^2-1)^j = t_0 + \sum_{j=1}^{\infty} c_j (K^2-1)^j,$$

where $c_j = \sum_{l=j}^{\infty} t_l b_{l,j}(v^2-1)^{l-j} \in \dot{\mathcal{A}} + \dot{\mathcal{A}}K$. Since $i_{\hat{\mathcal{U}}^0}(t) = 0$, we have $t_0 = 0$, and, for each $j \geq 1$, we have $c_j = 0$. Since $b_{j,j} = 1$, we can inductively verify that each t_n $(n \geq 1)$ is divisible by $(v^2-1)^m$ for all $m \geq 0$. We therefore have $t_1 = t_2 = \cdots = 0$, and consequently $t = 0$. Therefore $i_{\hat{\mathcal{U}}^0}$ is injective.

The other two cases $* = \geq 0$ and $* =$ (none) follow from the case $* = 0$, Proposition 5.7, Corollary 5.11, Proposition 6.5, and Proposition 6.6. □

In what follows, for $* = 0, \geq 0$, (none), we regard $\hat{\mathcal{U}}^*$ as an $\hat{\mathcal{A}}$-subalgebra of $\dot{\mathcal{U}}^*$ via $i_{\hat{\mathcal{U}}^*}$, and hence as an $\hat{\mathcal{A}}$-subalgebra of U_h^*.

Since $\dot{\mathcal{U}}^0 \subset U_h^0$ are integral domains, we have the following.

Corollary 6.10. *The ring $\hat{\mathcal{U}}^0$ is an integral domain.*

Since $\mathcal{U}_1^0$ is preserved by γ_j for each $j \in \mathbb{Z}$, there is an induced automorphism $\gamma_j : \dot{\mathcal{U}}^0 \to \dot{\mathcal{U}}^0$. In view of the inclusion $\hat{\mathcal{U}}^0 \subset \dot{\mathcal{U}}^0 \subset U_h^0$, the automorphism γ_j of $\hat{\mathcal{U}}^0$ and that of $\dot{\mathcal{U}}^0$ are restrictions of $\gamma_j : U_h^0 \to U_h^0$.

7. The filtration $\{\mathcal{U}_n^e\}_n$, the completion $\check{\mathcal{U}}$, and the subalgebra $\tilde{\mathcal{U}}$ of $\hat{\mathcal{U}}$

7.1. The filtration $\{\mathcal{U}_n^e\}_n$ of $\mathcal{U}$ and the completion $\check{\mathcal{U}}$ of $\mathcal{U}$

Recall from the introduction that, for $n \geq 0$, $\mathcal{U}_n^e$ denotes the two-sided ideal in $\mathcal{U}$ generated by the element e^n. We have

$$\mathcal{U}_n^e = \mathcal{U}e^n\mathcal{U} = \mathcal{U}\mathcal{U}_n^+\mathcal{U} = \mathcal{U}^-\mathcal{U}^0\mathcal{U}_n^+\mathcal{U}^- = \mathcal{U}^0\mathcal{U}^-\mathcal{U}_n^+\mathcal{U}^-. \tag{7.1}$$

Clearly, $\{\mathcal{U}_n^e\}_n$ is a decreasing filtration.

Proposition 7.1. *The family $\{\mathcal{U}_n^e\}_n$ is a fine Hopf algebra filtration of $\mathcal{U}$ over $\mathcal{A}$.*

Proof. The statement follows from (3.6) and (3.7). □

The completion $\check{\mathcal{U}} = \varprojlim_n \mathcal{U}/\mathcal{U}_n^e$, defined in the introduction, is a complete Hopf $\mathcal{A}$-algebra.

7.2. The image $\tilde{\mathcal{U}}$

We have $\mathcal{U}_n^e \subset \mathcal{U}_n$ for $n \geq 0$. Hence $\mathrm{id}_{\mathcal{U}}$ induces an $\mathcal{A}$-algebra homomorphism

$$i_{\check{\mathcal{U}}} : \check{\mathcal{U}} \to \hat{\mathcal{U}}.$$

Conjecture 7.2. *$i_{\check{\mathcal{U}}}$ is injective.*

Let $\tilde{\mathcal{U}}$ denote the image $i_{\check{\mathcal{U}}}(\check{\mathcal{U}}) \subset \hat{\mathcal{U}}$, which is an $\mathcal{A}$-subalgebra of $\hat{\mathcal{U}}$.

7.3. Hopf algebra structure of $\check{\mathcal{U}}$

If Conjecture 7.2 holds, then we can identify $\tilde{\mathcal{U}}$ and $\check{\mathcal{U}}$, and hence $\tilde{\mathcal{U}}$ has a complete Hopf algebra structure. Even if Conjecture 7.2 is not true, we can define a Hopf algebra structure on $\tilde{\mathcal{U}}$ in a suitable sense as follows. For $n \geq 0$, let $\tilde{\mathcal{U}}^{\tilde{\otimes} n}$ denote the image of the natural map

$$\check{\mathcal{U}}^{\check{\otimes} n} \to \hat{\mathcal{U}}^{\hat{\otimes} n}$$

between the completed tensor products. We have natural isomorphisms $\tilde{\mathcal{U}}^{\tilde{\otimes}0} \simeq \mathcal{A}$ and $\tilde{\mathcal{U}}^{\tilde{\otimes}1} \simeq \tilde{\mathcal{U}}$. The complete Hopf algebra structure of $\check{\mathcal{U}}$ induces continuous $\mathcal{A}$-module maps

$$\tilde{\Delta}\colon \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}^{\tilde{\otimes}2}, \quad \tilde{\epsilon}\colon \tilde{\mathcal{U}} \to \mathcal{A}, \quad \tilde{S}\colon \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}.$$

These maps satisfy the Hopf algebra axioms, such as coassociativity

$$(\tilde{\Delta} \otimes \mathrm{id})\tilde{\Delta} = (\mathrm{id} \otimes \tilde{\Delta})\tilde{\Delta}\colon \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}^{\tilde{\otimes}3}.$$

8. A $(\mathbb{Z}/2\mathbb{Z})$-grading of U and the induced gradings

Let $\mathcal{G}_0 U$ denote the $\mathbb{Q}(v)$-subalgebra of U generated by the elements K^2, K^{-2}, E, and FK. Set $\mathcal{G}_1 U = K\mathcal{G}_0 U$. Then we have a direct sum decomposition

$$U = \mathcal{G}_0 U \oplus \mathcal{G}_1 U, \tag{8.1}$$

which gives U a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Q}(v)$-algebra structure. An element of U is called *K-homogeneous* if it is homogeneous in the grading (8.1). Moreover, an additive subgroup M of U is called *K-homogeneous* if M is homogeneous in the grading (8.1), i.e., M is generated by K-homogeneous elements. For any K-homogeneous additive subgroup M of U and $i = 0, 1$, set

$$\mathcal{G}_i M = \mathcal{G}_i U \cap M.$$

It is easy to see that $\mathcal{U}$ is K-homogeneous. Hence (8.1) induces a $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathcal{A}$-algebra structure on $\mathcal{U}$:

$$\mathcal{U} = \mathcal{G}_0\mathcal{U} \oplus \mathcal{G}_1\mathcal{U},$$

and $\mathcal{G}_0\mathcal{U}$ is generated as an $\mathcal{A}$-algebra by the elements K^2, K^{-2}, e, and $F^{(i)}K^i$ for $i \geq 1$.

The two-sided ideals $\mathcal{U}_n$, $(\mathcal{U}_1)^n$, and $\mathcal{U}_n^e$ for $n \geq 0$ are all K-homogeneous since their generators are K-homogeneous. Therefore $\vec{\mathcal{U}} = \hat{\mathcal{U}}, \dot{\mathcal{U}}, \check{\mathcal{U}}, \tilde{\mathcal{U}}$ inherits from $\mathcal{U}$ a $(\mathbb{Z}/2\mathbb{Z})$-graded algebra structure

$$\vec{\mathcal{U}} = \mathcal{G}_0\vec{\mathcal{U}} \oplus \mathcal{G}_1\vec{\mathcal{U}}.$$

The 0th graded part $\mathcal{G}_0\vec{\mathcal{U}}$ is naturally identified with a completion of $\mathcal{G}_0\mathcal{U}$ as follows:

$$\mathcal{G}_0\hat{\mathcal{U}} \simeq \varprojlim_n \mathcal{G}_0\mathcal{U}/\mathcal{G}_0\mathcal{U}_n, \quad \mathcal{G}_0\dot{\mathcal{U}} \simeq \varprojlim_n \mathcal{G}_0\mathcal{U}/\mathcal{G}_0(\mathcal{U}_1)^n,$$

$$\mathcal{G}_0\check{\mathcal{U}} \simeq \varprojlim_n \mathcal{G}_0\mathcal{U}/\mathcal{G}_0\mathcal{U}_n^e, \quad \mathcal{G}_0\tilde{\mathcal{U}} \simeq i_{\check{\mathcal{U}}}(\mathcal{G}_0\check{\mathcal{U}}).$$

9. Centers

9.1. The Harish–Chandra homomorphism for U_h

Let

$$\varphi\colon \Gamma_0 U_h \to U_h^0$$

denote the *Harish–Chandra homomorphism* of U_h, which is defined to be the continuous $\mathbb{Q}[[h]]$-module homomorphism satisfying

$$\varphi(F^i t E^i) = \delta_{i,0} t$$

for $i \geq 0$ and $t \in U_h^0$. Set

$$w = \gamma_2 S = S\gamma_{-2}\colon U_h^0 \to U_h^0,$$

which is the (involutive) continuous $\mathbb{Q}[[h]]$-algebra automorphism of U_h^0 determined by $w(H) = -H - 2$. Set

$$(U_h^0)^w = \{t \in U_h^0 \mid w(t) = t\}.$$

We will use the following.

Theorem 9.1 (*[14]*). *The homomorphism φ maps the center $Z(U_h)$ isomorphically onto $(U_h^0)^w$.*

9.2. Facts about the centers of $\mathcal{U}$, $\tilde{\mathcal{U}}$, $\hat{\mathcal{U}}$, and $\dot{\mathcal{U}}$

In the rest of this section we will determine the centers of the subalgebras $\mathcal{U} \subset \tilde{\mathcal{U}} \subset \hat{\mathcal{U}} \subset \dot{\mathcal{U}}$ of U_h. Let $\vec{\mathcal{U}}$ denote one of these algebras. We have $Z(\vec{\mathcal{U}}) = \vec{\mathcal{U}} \cap Z(U_h)$ since if $z \in \vec{\mathcal{U}}$ is central in $\vec{\mathcal{U}}$, then z commutes with K, F, e and hence also with H and E, and consequently z is central in U_h.

If $\vec{\mathcal{U}} = \mathcal{U}, \hat{\mathcal{U}}$, or $\dot{\mathcal{U}}$, then let $\vec{\mathcal{U}}^0$ denote $\mathcal{U}^0$, $\hat{\mathcal{U}}^0$, or $\dot{\mathcal{U}}^0$, respectively. In view of Proposition 3.1, Corollary 5.11, and Proposition 6.6, we have

$$\vec{\mathcal{U}}^0 = \varphi(\Gamma_0 \vec{\mathcal{U}}).$$

If $\vec{\mathcal{U}} = \tilde{\mathcal{U}}$, then set

$$\vec{\mathcal{U}}^0 = \tilde{\mathcal{U}}^0 = \varphi(\Gamma_0 \tilde{\mathcal{U}}) \subset \hat{\mathcal{U}}^0.$$

(*Remark*: the notation $\tilde{\mathcal{U}}^0$ might suggest that $\tilde{\mathcal{U}}^0 = \tilde{\mathcal{U}} \cap \hat{\mathcal{U}}^0$, but this is not the case. In fact, $\tilde{\mathcal{U}}^0$ is not contained in $\tilde{\mathcal{U}}$.)

For $\vec{\mathcal{U}} = \mathcal{U}, \tilde{\mathcal{U}}, \hat{\mathcal{U}}, \dot{\mathcal{U}}$, set

$$(\vec{\mathcal{U}}^0)^w = \vec{\mathcal{U}}^0 \cap (U_h)^w = \{t \in \vec{\mathcal{U}}^0 \mid w(t) = t\}.$$

In view of Theorem 9.1, φ maps $Z(\vec{\mathcal{U}})$ injectively into $(\vec{\mathcal{U}}^0)^w$.

9.3. Center of $\mathcal{U}$

It is well known (see e.g. [7, Proposition 2.18]) that the center $Z(U)$ is freely generated as a $\mathbb{Q}(v)$-algebra by the element

$$C = fe + vK + vK^{-1} = (v - v^{-1})Fe + vK + v^{-1}K^{-1} \in Z(\overline{U}) \subset Z(\mathcal{U}),$$

i.e., we have

$$Z(U) = \mathbb{Q}(v)[C].$$

Theorem 9.2. *The center $Z(\mathcal{U})$ of $\mathcal{U}$ is freely generated as an $\mathcal{A}$-algebra by the element C, i.e., we have*

$$Z(\mathcal{U}) = \mathcal{A}[C]. \tag{9.1}$$

The Harish–Chandra homomorphism φ maps $Z(\mathcal{U})$ isomorphically onto $(\mathcal{U}^0)^w$. We also have

$$Z(\mathcal{U}) = Z(\overline{U}).$$

Proof. Since $w(K^{\pm 1}) = v^{\mp 2}K^{\mp 1} \in \mathcal{U}$, the automorphism w of U_h^0 restricts to an automorphism $w\colon \mathcal{U}^0 \to \mathcal{U}^0$. One can verify that the subalgebra $(\mathcal{U}^0)^w$ is freely generated as an $\mathcal{A}$-algebra by the element $\varphi(C) = vK + vK^{-1}$. By Section 9.2, $Z(\mathcal{U})$ is mapped injectively into $(\mathcal{U}^0)^w$, and hence we have $Z(\mathcal{U}) \subset \mathcal{A}[C]$. Since $C \in Z(\mathcal{U})$, we have $\mathcal{A}[C] \subset Z(\mathcal{U})$. Therefore (9.1) holds and $\varphi(Z(U)) = (\mathcal{U}^0)^w$. The last statement follows from (9.1) and $C \in Z(\overline{U})$. □

Since $C \in \mathcal{G}_1(Z(\mathcal{U}))$, it follows that the center $Z(\mathcal{U})$ is K-homogeneous and

$$\mathcal{G}_0 Z(\mathcal{U}) = \mathcal{A}[C^2], \quad \mathcal{G}_1 Z(\mathcal{U}) = C\mathcal{G}_0 Z(\mathcal{U}) = C\mathcal{A}[C^2].$$

9.4. A basis of $Z(\mathcal{U})$

As in the introduction, we set for $n \geq 0$

$$\sigma_n = \prod_{i=1}^{n}(C^2 - (v^i + v^{-i})^2) \in Z(\overline{U}) \subset Z(\mathcal{U}).$$

Note that the σ_n for $n \geq 0$ form a basis of $\mathcal{G}_0 Z(\mathcal{U})$, and hence the $C\sigma_n$ for $n \geq 0$ form a basis of $\mathcal{G}_1 Z(\mathcal{U})$.

Lemma 9.3. *For $n \geq 0$,*

$$Z(\mathcal{U}) \cap U E^n U \subset \sigma_n Z(\mathcal{U}).$$

Proof. For $i \geq 1$, there is an i-dimensional irreducible left U-module $V_i^{\pm}$ generated by a highest weight vector $u_i^{\pm}$ satisfying $K u_i^{\pm} = \pm v^{i-1} u_i^{\pm}$. It is known that C acts on $V_i^{\pm}$ by the scalar $\pm(v^i + v^{-i})$.

Suppose that $z \in Z(\mathcal{U}) \cap U E^n U$. Write z as a polynomial $z = g(C) \in \mathcal{A}[C]$. Since $z \in U E^n U$, z acts as 0 on the i-dimensional irreducible representations V_i^+ and V_i^- with $i \leq n$. Since z acts on $V_i^{\pm}$ by the scalar $g(\pm(v^i + v^{-i}))$, we have $g(\pm(v^i + v^{-i})) = 0$ for $i = 1, \ldots, n$. Hence the polynomial $z = g(C)$ is divisible by

$$\prod_{i=1}^{n}(C - (v^i + v^{-i}))(C + (v^i + v^{-i})) = \prod_{i=1}^{n}(C^2 - (v^i + v^{-i})^2) = \sigma_n. \quad \square$$

The following is proved in Section 10.

Proposition 9.4. *For each $n \geq 0$, we have $\sigma_n \in \mathcal{U}_n^e$.*

Assuming Proposition 9.4, we have the following.

Theorem 9.5. *For each $n \geq 0$,*

$$\sigma_n Z(\mathcal{U}) = Z(\mathcal{U}) \cap \mathcal{U}_n^e = Z(\mathcal{U}) \cap U E^n U.$$

Proof. Lemma 9.3 and Proposition 9.4 imply that

$$Z(\mathcal{U}) \cap U E^n U \subset \sigma_n Z(\mathcal{U}) \subset Z(\mathcal{U}) \cap \mathcal{U}_n^e \subset Z(\mathcal{U}) \cap U E^n U,$$

where the last inclusion is obvious. Hence the statement follows. $\square$

9.5. Center of $\hat{\mathcal{U}}$

For $n \geq 0$, we have $w(\mathcal{U}_n^0) = \gamma_2 S(\mathcal{U}_n^0) = \gamma_2(\mathcal{U}_n^0) = \mathcal{U}_n^0$. Hence w restricts to an involutive, continuous $\hat{\mathcal{A}}$-algebra automorphism of $\hat{\mathcal{U}}^0$. For $n \geq 0$, set

$$\bar{\sigma}_n = \varphi(\sigma_n) = \prod_{i=1}^{n}(\varphi(C)^2 - (v^i + v^{-i})^2) = \{H\}_n\{H + 1 + n\}_n.$$

Lemma 9.6. *We have $\hat{\mathcal{U}}^0 \simeq \varprojlim_n \hat{\mathcal{A}}[K, K^{-1}]/(\bar{\sigma}_n)$.*

Proof. Set $(\hat{\mathcal{U}}^0)' = \varprojlim_n \hat{\mathcal{A}}[K, K^{-1}]/(\bar{\sigma}_n)$. Since $(\bar{\sigma}_n) \subset (\{H\}_n)$, $\mathrm{id}_{\hat{\mathcal{A}}[K,K^{-1}]}$ induces a continuous $\hat{\mathcal{A}}$-algebra homomorphism $f_1\colon (\hat{\mathcal{U}}^0)' \to \hat{\mathcal{U}}^0$. We will show that there is a natural homomorphism $f_2\colon \hat{\mathcal{U}}^0 \to (\hat{\mathcal{U}}^0)'$. By an argument similar to the one used in the proof of Proposition 5.4, we see that $(\hat{\mathcal{U}}^0)' \simeq \varprojlim_{m,n} \mathcal{A}[K, K^{-1}]/(\{m\}!, \bar{\sigma}_n)$. Hence it suffices to prove that, for each $m, n \geq 0$, there is $n' \geq 0$ such that $\{H\}_{n'} \in (\{m\}!, \bar{\sigma}_n)$. By Proposition 5.1, $\mathcal{U}_{4n}^0 \subset (\{H\}_{2n+1}, \{2n\}!)$. Applying the homomorphism γ_{n+1} to both sides, we obtain $\mathcal{U}_{4n}^0 \subset (\{H+n+1\}_{2n+1}, \{2n\}!) \subset (\bar{\sigma}_n, \{2n\}!)$. Hence, by Proposition 5.1, we have $\{H\}_{n'} \in (\{m\}!, \bar{\sigma}_n)$ where $n' = \max(4n, 2m)$. Hence f_2 exists. Clearly, f_2 is inverse to f_1. $\square$

Remark 9.7. By the proof of Lemma 9.6, we have

$$\hat{\mathcal{U}}^0 \simeq \varprojlim_n \hat{\mathcal{A}}[K, K^{-1}]/(\{H+n+1\}_{2n+1}).$$

Therefore

$$\hat{\mathcal{U}}^0 \simeq \varprojlim_{X\subset\mathbb{Z},\ |X|<\infty} \hat{\mathcal{A}}[K, K^{-1}]/\left(\prod_{k\in X}\{H+k\}\right),$$

where X runs through all the finite subsets of $\mathbb{Z}$.

Since $w\colon \hat{\mathcal{U}}^0 \to \hat{\mathcal{U}}^0$ preserves the $(\mathbb{Z}/2\mathbb{Z})$-grading $\hat{\mathcal{U}}^0 = \mathcal{G}_0\hat{\mathcal{U}}^0 \oplus \mathcal{G}_1\hat{\mathcal{U}}^0$, it follows that $(\hat{\mathcal{U}}^0)^w$ is K-homogeneous and hence has a $(\mathbb{Z}/2\mathbb{Z})$-graded algebra structure: $(\hat{\mathcal{U}}^0)^w = \mathcal{G}_0(\hat{\mathcal{U}}^0)^w \oplus \mathcal{G}_1(\hat{\mathcal{U}}^0)^w$.

Lemma 9.8. *Each element of $\mathcal{G}_0(\hat{\mathcal{U}}^0)^w$ is uniquely expressed as an infinite sum $t = \sum_{n\geq 0} t_n\bar{\sigma}_n$, where $t_n \in \hat{\mathcal{A}}$ for $n \geq 0$. We have $\mathcal{G}_1(\hat{\mathcal{U}}^0)^w = \varphi(C)\mathcal{G}_0(\hat{\mathcal{U}}^0)^w$.*

Proof. By Lemma 9.6, each element $t \in \mathcal{G}_0\hat{\mathcal{U}}^0$ is uniquely expressed as

$$t = \sum_{n\geq 0}(t_n + t'_nK^2)\bar{\sigma}_n$$

where $t_n, t'_n \in \hat{\mathcal{A}}$, $n \geq 0$. Suppose that $t \in \mathcal{G}_0(\hat{\mathcal{U}}^0)^w$. It suffices to show that $t'_n = 0$ for $n \geq 0$. We may assume without loss of generality that $t_n = 0$ for each $n \geq 0$, and hence $t = \sum_n t'_nK^2\bar{\sigma}_n$. The proof is by induction on n. Suppose that $t_0 = t_1 = \cdots = t_{n-1} = 0$ for $n \geq 0$. Then $t \equiv t'_nK^2\bar{\sigma}_n (\mathrm{mod}\ (\bar{\sigma}_{n+1}))$. Since $t = w(t)$, we have $t'_nK^2\bar{\sigma}_n \equiv t'_nv^{-4}K^{-2}\bar{\sigma}_n (\mathrm{mod}\ (\bar{\sigma}_{n+1}))$, and hence $t'_n(K^2 - v^{-4}K^{-2}) \in (\{H-n\}\{H+n+2\})$. Since $K^2 - v^{-4}K^{-2}$ is not divisible by $\{H-n\}\{H+n+2\}$, we have $t'_n = 0$. This proves the first statement.

Suppose that $t \in \mathcal{G}_1(\hat{\mathcal{U}}^0)^w$. By Lemma 9.6, t is uniquely expressed as an infinite sum $t = \sum_n t_n\bar{\sigma}_n$, where $t_0, t_1, \ldots \in \hat{\mathcal{A}}K + \hat{\mathcal{A}}K^{-1}$. Then $w(t) = \sum_n w(t_n)\bar{\sigma}_n$, and $w(t_n) \in \hat{\mathcal{A}}K + \hat{\mathcal{A}}K^{-1}$ for $n \geq 0$. Since $w(t) = t$, we have $\sum_n(t_n - w(t_n))\bar{\sigma}_n = 0$, and hence $t_n = w(t_n)$ for $n \geq 0$. Therefore $t_n \in \hat{\mathcal{A}}\varphi(C)$ for all $n \geq 0$, and hence $t \in \varphi(C)\mathcal{G}_0(\hat{\mathcal{U}}^0)^w$. This proves the second statement. □

Theorem 9.9. *Each element $z \in Z(\hat{\mathcal{U}})$ is uniquely expressed as an infinite sum $z = \sum_{n\geq 0} z_n\sigma_n$, where $z_n \in \hat{\mathcal{A}} + \hat{\mathcal{A}}C$ for $n \geq 0$. The Harish–Chandra homomorphism φ maps $Z(\hat{\mathcal{U}})$ isomorphically onto $(\hat{\mathcal{U}}^0)^w$. We have*

$$Z(\hat{\mathcal{U}}) \simeq \varprojlim_{m,n} Z(\mathcal{U})/(\{m\}!, \sigma_n) \simeq \varprojlim_n \hat{\mathcal{A}}[C]/(\sigma_n).$$

The center $Z(\hat{\mathcal{U}})$ is K-homogeneous, and we have

$$\mathcal{G}_0Z(\hat{\mathcal{U}}) \simeq \varprojlim_n \hat{\mathcal{A}}[C^2]/(\sigma_n), \quad \mathcal{G}_1Z(\hat{\mathcal{U}}) = C\mathcal{G}_0Z(\hat{\mathcal{U}}).$$

Proof. By Section 9.2, φ maps $Z(\hat{\mathcal{U}})$ injectively into $(\hat{\mathcal{U}}^0)^w$. By Lemma 9.8, each element t of $(\hat{\mathcal{U}}^0)^w$ is uniquely expressed as an infinite sum $t = \sum_{n\geq 0}(t_n + t'_n\varphi(C))\bar{\sigma}_n$, where $t_n, t'_n \in \hat{\mathcal{A}}$ for $n \geq 0$. The element $z = \sum_{n\geq 0}(t_n + t'_nC)\sigma_n$ is contained in $Z(\hat{\mathcal{U}})$ since we have $\sigma_n \in \mathcal{U}_n$ by Proposition 9.4. Obviously, $\varphi(z) = t$. Hence the first and the second statements follow. The rest of the theorem easily follows. □

9.6. Center of $\tilde{\mathcal{U}}$

Lemma 9.10. *If $n \geq 0$, then $\varphi(\Gamma_0\mathcal{U}^e_n) = (\{H\}_n) \subset \mathcal{U}^0$. Consequently, each element of $\tilde{\mathcal{U}}^0$ is uniquely expressed as an infinite sum $t = \sum_{n\geq 0} t_n\{H\}_n$, where $t_n \in \mathcal{A}$ for $n \geq 0$.*

Proof. In view of (7.1), $\Gamma_0\mathcal{U}^e_n$ is $\mathcal{A}$-spanned by the elements

$$tF^{(i)}e^mF^{(k)}e^l \quad (i, k, l \geq 0, m \geq n, m + l = i + k, t \in \mathcal{U}^0).$$

Since $\varphi(tF^{(i)}e^mF^{(k)}e^l) = 0$ if either $i > 0$ or $l > 0$, it follows that $\varphi(\Gamma_0\mathcal{U}_n^e)$ is $\mathcal{A}$-spanned by the elements $t\varphi(e^mF^{(m)})$ for $m \geq n$ and $t \in \mathcal{U}^0$. By (3.1), we have $\varphi(e^mF^{(m)}) = \{H\}_m$. Hence the first statement follows. □

Lemma 9.11. *Each element of $\mathcal{G}_0(\tilde{\mathcal{U}}^0)^w$ is uniquely expressed as an infinite sum $t = \sum_{n\geq 0} t_n\bar{\sigma}_n$, where $t_n \in \mathcal{A}$ for $n \geq 0$. We have $\mathcal{G}_1(\tilde{\mathcal{U}}^0)^w = \varphi(C)\mathcal{G}_0(\tilde{\mathcal{U}}^0)^w$.*

Proof. For $j \in \mathbb{Z}$, let $s_j : \hat{\mathcal{U}} \to \hat{\mathcal{A}}$ denote the continuous $\hat{\mathcal{A}}$-algebra homomorphism determined by $s_j(K) = v^j$.

Let t be an element of $\mathcal{G}_0(\tilde{\mathcal{U}}^0)^w$ (resp. $\mathcal{G}_1(\tilde{\mathcal{U}}^0)^w$). By Lemma 9.8, t is uniquely expressed as $t = \sum_{n\geq 0} t_n\bar{\sigma}_n = \sum_{n\geq 0} t_n\{H\}_n\{H+n+1\}_n$, where $t_n \in \hat{\mathcal{A}}$ (resp. $t_n \in \hat{\mathcal{A}}\varphi(C)$) for $n \geq 0$. We prove by induction on n that $t_n \in \mathcal{A}$ (resp. $t_n \in \mathcal{A}\varphi(C)$) for $n \geq 0$. Let $n \geq 0$ and assume that $t_i \in \mathcal{A}$ (resp. $t_i \in \mathcal{A}\varphi(C)$) if $0 \leq i < n$. By Lemma 9.10, we have $s_n(t) \in \mathcal{A}$, and hence $\mathcal{A} \ni s_n(t) = \sum_{i=0}^{n} s_n(t_i)\{n\}_i\{n+i+1\}_i$. By the inductive hypothesis, $s_n(t_n)\{n\}_n\{2n+1\}_n \in \mathcal{A}$. In the case $t \in \mathcal{G}_0(\tilde{\mathcal{U}}^0)^w$, we see using Lemma 9.12 below that $s_n(t_n) = t_n \in \mathcal{A}$. In the case $t \in \mathcal{G}_1(\tilde{\mathcal{U}}^0)^w$, we have $t_n = b_n\varphi(C)$ for some $b_n \in \hat{\mathcal{A}}$. Hence $s_n(t_n) = b_n(v^{n+1} + v^{-n-1}) \in \mathcal{A}$. Since $v^{n+1} + v^{-n-1}$ is equal up to multiplication by a power of v to the product of some cyclotomic polynomials in $q = v^2$, it follows from Lemma 9.12 below that $b_n \in \mathcal{A}$, and therefore $t_n \in \mathcal{A}\varphi(C)$. □

For $i \geq 1$, let $\Phi_i(q) \in \mathbb{Z}[q] \subset \mathcal{A}_q$ denote the ith cyclotomic polynomial in q.

Lemma 9.12. *Suppose that $t \in \hat{\mathcal{A}}$ and $t\Phi(q) \in \mathcal{A}$, where $\Phi(q) = \prod_{i\in\mathbb{N}} \Phi_i(q)^{\lambda(i)} \in \mathbb{Z}[q]$ is the product of some powers of cyclotomic polynomials in $q = v^2$. Then $t \in \mathcal{A}$.*

Proof. If $t \in \hat{\mathcal{A}}_q$, then it follows from [3, Proposition 7.3] that $t \in \mathcal{A}_q$. If $t \in v\hat{\mathcal{A}}_q$, then the first case implies that $t \in v\mathcal{A}_q$. Then the general case immediately follows. □

Theorem 9.13. *Each element $z \in Z(\tilde{\mathcal{U}})$ is uniquely expressed as an infinite sum $z = \sum_{n\geq 0} z_n\sigma_n$, where $z_n \in \mathcal{A}+\mathcal{A}C$ for $n \geq 0$. The Harish–Chandra homomorphism φ maps $Z(\tilde{\mathcal{U}})$ isomorphically onto $(\tilde{\mathcal{U}}^0)^w$. We have*

$$Z(\tilde{\mathcal{U}}) \simeq \varprojlim_n Z(\mathcal{U})/(\sigma_n).$$

The center $Z(\tilde{\mathcal{U}})$ is K-homogeneous, and we have

$$\mathcal{G}_0 Z(\tilde{\mathcal{U}}) \simeq \varprojlim_n \mathcal{A}[C^2]/(\sigma_n), \quad \mathcal{G}_1 Z(\tilde{\mathcal{U}}) = C\mathcal{G}_0 Z(\tilde{\mathcal{U}}).$$

Proof. Let $z \in Z(\tilde{\mathcal{U}})$. By Lemma 9.11, we have $\varphi(z) = \sum_{n\geq 0}(a_n + a'_n\varphi(C))\bar{\sigma}_n$ with $a_n, a'_n \in \mathcal{A}$ for $n \geq 0$. Set $z' = \sum_{n\geq 0}(a_n + a'_nC)\sigma_n$, which is an element of $Z(\tilde{\mathcal{U}})$ by Proposition 9.4, and we have $\varphi(z) = \varphi(z')$. Since φ is injective, we have $z = z'$. Therefore the first and the second statements follow. The rest of the theorem follows from the first statement. □

9.7. Center of $\dot{\mathcal{U}}$

For completeness, we state the following results.

Theorem 9.14. *Each element $z \in Z(\dot{\mathcal{U}})$ is uniquely expressed as a formal power series $z = \sum_{n\geq 0} z_n(C^2 - [2]^2)^n$, where $z_n \in \dot{\mathcal{A}} + \dot{\mathcal{A}}C$ for $n \geq 0$. (Note that $C^2 - [2]^2 = \sigma_1$.) The Harish–Chandra homomorphism φ maps $Z(\dot{\mathcal{U}})$ isomorphically onto $(\dot{\mathcal{U}}^0)^w$. We have*

$$Z(\dot{\mathcal{U}}) \simeq \varprojlim_{m,n} Z(\mathcal{U})/(\{1\}^m, (C^2 - [2]^2)^n) \simeq \varprojlim_n \dot{\mathcal{A}}[C]/((C^2 - [2]^2)^n),$$

$$\mathcal{G}_0 Z(\dot{\mathcal{U}}) \simeq \dot{\mathcal{A}}[[C^2 - [2]^2]], \quad \mathcal{G}_1 Z(\dot{\mathcal{U}}) = C\mathcal{G}_0 Z(\dot{\mathcal{U}}).$$

The proof of Theorem 9.14 is similar to that of Theorem 9.9. One also has a version of Theorem 9.14 in which the occurrences of $(C^2 - [2]^2)^n$ are replaced with σ_n as follows.

Theorem 9.15. *Each element $z \in Z(\dot{\mathcal{U}})$ is uniquely expressed as an infinite sum $z = \sum_{n\geq 0} z_n\sigma_n$, where $z_n \in \dot{\mathcal{A}}+\dot{\mathcal{A}}C$ for $n \geq 0$. We have*

$$Z(\dot{\mathcal{U}}) \simeq \varprojlim_{m,n} Z(\mathcal{U})/(\{1\}^m, \sigma_n) \simeq \varprojlim_{n} \dot{\mathcal{A}}[C]/(\sigma_n).$$

Proof. Use Theorem 9.14 and the fact that the double filtrations $\{(\{1\}^m, \sigma_n)\}_{m,n}$ and $\{(\{1\}^m, (C^2 - [2]^2)^n)\}_{m,n}$ of $Z(\mathcal{U})$ are cofinal with each other. □

9.8. Multiplication in $Z(\mathcal{U})$, $Z(\hat{\mathcal{U}})$, and $Z(\tilde{\mathcal{U}})$

Multiplication in the centers $Z(\mathcal{U})$, $Z(\hat{\mathcal{U}})$, and $Z(\tilde{\mathcal{U}})$ is described by giving a formula for the product $\sigma_m\sigma_n$ as a linear combination of the σ_j.

Proposition 9.16. *If $m, n \geq 0$, then*

$$\begin{aligned}\sigma_m\sigma_n &= \sum_{i=0}^{\min(m,n)} \{m\}_i\{n\}_i \begin{bmatrix} m+n+1 \\ i \end{bmatrix} \sigma_{m+n-i} \\ &= \sum_{j=\max(m,n)}^{m+n} \{m\}_{m+n-j}\{n\}_{m+n-j} \begin{bmatrix} m+n+1 \\ j+1 \end{bmatrix} \sigma_j.\end{aligned}$$

Proof. The proof is a straightforward induction on n using the identity

$$\sigma_m\sigma_{n+1} = \sigma_{m+1}\sigma_n + \{m-n\}\{m+n+2\}\sigma_m\sigma_n. \quad \square$$

Remark 9.17. Suppose that $a = \sum_{m\geq 0} a_m\sigma_m$ and $b = \sum_{n\geq 0} b_n\sigma_n$ are elements of $\mathcal{G}_0Z(\hat{\mathcal{U}})$ (or $\mathcal{G}_0Z(\tilde{\mathcal{U}})$, $\mathcal{G}_0Z(\mathcal{U})$); then

$$ab = \sum_{j\geq 0}\left(\sum_{0\leq m,n\leq j;\ m+n\geq j} \{m\}_{m+n-j}\{n\}_{m+n-j} \begin{bmatrix} m+n+1 \\ j+1 \end{bmatrix} a_m b_n\right)\sigma_j.$$

10. Proof of Proposition 9.4

In this section, we will prove Proposition 9.4. In the first two Sections 10.1 and 10.2, we prepare necessary facts. The proof is given in Section 10.3, in which we need a result in Section 10.4.

10.1. Formulas in $\mathcal{U}^0$

For $a \in \mathbb{Z}H + \mathbb{Z}$ and $r \geq 0$, we have the following well-known formula:

$$\{a\}_r = \sum_{j=0}^{r} (-1)^j v^{\frac{1}{2}(r-2j)(-r+1+2a)} \begin{bmatrix} r \\ j \end{bmatrix}, \tag{10.1}$$

which one can verify by induction on r.

For $r, s \geq 0$, $a, b, c \in \mathbb{Z}H + \mathbb{Z}$, set

$$\begin{aligned}\kappa(a, r; b, s; c) &= \sum_{i=0}^{r}\sum_{j=0}^{s} (-1)^{i+j} v^{\frac{1}{2}(r-2i)(-r+1+2a)+\frac{1}{2}(s-2j)(-s+1+2b)+\frac{1}{2}c(rs-4ji)} \begin{bmatrix} r \\ i \end{bmatrix}\begin{bmatrix} s \\ j \end{bmatrix} \\ &= \sum_{j=0}^{s} (-1)^j v^{\frac{1}{2}(s-2j)(-s+1+2b+rc)} \begin{bmatrix} s \\ j \end{bmatrix} \{a+jc\}_r.\end{aligned}$$

where the identity follows from (10.1). We have the symmetry

$$\kappa(a, r; b, s; c) = \kappa(b, s; a, r; c). \tag{10.2}$$

Note that $\kappa(a, r; b, s; c) \in v^{\frac{1}{2}crs}\mathcal{U}^0$.

Lemma 10.1. *If $0 \le j \le m$, then*

$$\sum_{i=j}^{m} (-1)^i v^{i(j-m)} \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} m+1 \\ i+1 \end{bmatrix} = (-1)^j v^{m-j}. \tag{10.3}$$

Proof. It is well known that

$$\sum_{i=k}^{l} (-1)^i v^{i(k-l)} \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} \begin{bmatrix} l \\ i \end{bmatrix} = (-1)^k \tag{10.4}$$

for $1 \le k \le l$, which can also be derived from (10.2) by setting $(a, r, b, s, c) = (-1, k-1, l-k, l, 1)$. Formula (10.3) follows from (10.4) by setting $l = m+1$, $k = j+1$, and shifting i by 1. □

Lemma 10.2. *If $n \in \mathbb{Z}$ and $l \ge 0$, then*

$$\sum_{j=0}^{l} (-1)^j v^{j(l-1-n)} \begin{bmatrix} l \\ j \end{bmatrix} \{H-n\}_{l-j}\{H+j-l\}_j = (-1)^l v^{l^2-l}\{n\}_l K^{-l}. \tag{10.5}$$

Proof. Let λ_l and λ'_l denote the left and right hand sides of (10.5), respectively. It is straightforward to verify that the λ_n and the λ'_n satisfy the same recurrence relations:

$$\lambda_{l+1} = \{H-n\}\gamma_{-1}(\lambda_l) - v^{2l-n}\{H-l\}\lambda_l,$$
$$\lambda'_{l+1} = \{H-n\}\gamma_{-1}(\lambda'_l) - v^{2l-n}\{H-l\}\lambda'_l.$$

An induction proves the statement. □

10.2. Adjoint action

Let $\triangleright : U \otimes U \to U$, $x \otimes y \mapsto x \triangleright y$, denote the (left) adjoint action defined by

$$x \triangleright y = \sum x_{(1)} y S(x_{(2)})$$

for $x, y \in U$, where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. We will regard U as a left U-module by the adjoint action.

For a homogeneous element x in U and $m \ge 0$, we have

$$e^m \triangleright x = \sum_{i=0}^{m} (-1)^i v^{i(m-1+2|x|)} \begin{bmatrix} m \\ i \end{bmatrix} e^{m-i} x e^i, \tag{10.6}$$

$$F^{(m)} \triangleright x = \sum_{i=0}^{m} (-1)^i v^{-i(m-1)} F^{(m-i)} x F^{(i)} K^m, \tag{10.7}$$

which can easily be derived using formulas in Section 3.2.

For each $n \ge 0$, set

$$M_n = U \triangleright K^{-n}E^{(n)} = U \triangleright K^{-n}e^n \subset U,$$

which is an irreducible $(2n+1)$-dimensional left U-module, with a highest weight vector $K^{-n}E^{(n)}$ of weight v^{2n}, i.e., we have $E \triangleright K^{-n}E^{(n)} = 0$ and $K \triangleright K^{-n}E^{(n)} = v^{2n}$. We have

$$F^{(2n)} \triangleright K^{-n}E^{(n)} = (-1)^n v^{-n(n-1)} F^{(n)},$$

and $F^{(k)} \triangleright K^{-n}E^{(n)} = 0$ if $k > 2n$. The element $F^{(n)}$ is a lowest weight vector, i.e., $F \triangleright F^{(n)} = 0$.

Lemma 10.3. *If $n, j \geq 0$, then*

$$e^j \triangleright F^{(n)} = \sum_{k=0}^{n} v^{-\frac{1}{2}(j-k)(-j+1+2n)} \begin{bmatrix} j \\ k \end{bmatrix} \beta_{n,j,k} F^{(n-k)} e^{j-k},$$

where

$$\begin{aligned}\beta_{n,j,k} &= \kappa(n-k, j-k; H-j+n, k; 1) \\ &= \sum_{i=0}^{j-k} (-1)^i v^{\frac{1}{2}(j-k-2i)(-j+1+2n)} \begin{bmatrix} j-k \\ i \end{bmatrix} \{H-j+i+n\}_k \\ &= \sum_{l=0}^{k} (-1)^l v^{\frac{1}{2}(k-2l)(1-j+2n-2k)} \begin{bmatrix} k \\ l \end{bmatrix} \{n-k+l\}_{j-k} K^{k-2l}.\end{aligned}$$

Proof. The proof is straightforward. □

10.3. Proof of Proposition 9.4

For each $m \geq 0$, set

$$\xi_m = \prod_{i=1}^{m} (C - v^{2i+1} - v^{-2i-1}) \in Z(\overline{U}),$$
$$\xi'_m = \prod_{i=1}^{m} (C - v^{2i+1}K - v^{-2i-1}K^{-1}) \in \Gamma_0\overline{U}.$$

By induction on m, we can check that

$$\xi'_m = \{m\}! \sum_{i=0}^{m} (-1)^i \{H+m+1\}_i F^{(m-i)} e^{m-i}. \tag{10.8}$$

For $n \geq 0$, set

$$\widetilde{\sigma}_n = (\{2n\}!)^{-1} \xi_{2n} \triangleright F^{(n)} K^{-n} e^n = (\{2n\}!)^{-1} \xi'_{2n} \triangleright F^{(n)} K^{-n} e^n.$$

Here the identity holds since $K^{\pm 1} \triangleright x = x$ for any $x \in \Gamma_0\mathcal{U}$. In view of (10.8), we have $\widetilde{\sigma}_n \in \mathcal{U}^e_n$.

Lemma 10.4. *For each $n \geq 0$, we have $\widetilde{\sigma}_n \in \sigma_n Z(\mathcal{U})$.*

Proof. Due to Lemma 9.3, since $\widetilde{\sigma}_n \in \mathcal{U}^e_n$, it suffices to prove that $\widetilde{\sigma}_n$ is central. The elements $F^{(n)}$ and $K^{-n}e^n$ are contained in the $(2n+1)$-dimensional, irreducible left U-submodule $M_n = U \triangleright K^{-n}E^{(n)}$ of the adjoint representation of U. Hence the product $K^{-n}e^n$ is contained in $M_n M_n = \mu(M_n \otimes M_n) \subset U$, where $\mu \colon U \otimes U \to U$ is the multiplication. Since M_n is a $(2n+1)$-dimensional irreducible representation, it follows from the Clebsch–Gordan rule that $M_n \otimes M_n$ has a direct sum decomposition

$$M_n \otimes M_n = W_0 \oplus W_1 \oplus \cdots \oplus W_{2n},$$

where, for each $i = 0, \ldots, 2n$, W_i is the $(2n+1)$-dimensional irreducible U-module generated by a highest weight vector of weight v^{2n}. Therefore

$$M_n M_n = W'_0 \oplus W'_1 \oplus \cdots \oplus W'_{2n},$$

where $W'_i = \mu(W_i)$ is isomorphic to either W_i or zero for each $i = 0, \ldots, 2n$. (In fact, all the W'_i are nonzero, but we will not need this fact.) Since, for each $i = 1, \ldots, m$, the element $C - v^{2i+1} - v^{-2i-1}$ acts as 0 on W_i, the element ξ_{2n} acts by the adjoint action as 0 on W'_i. Therefore we have $\widetilde{\sigma}_n \in W'_0$, and hence $\widetilde{\sigma}_n$ is central. □

Lemma 10.5. *For $n \geq 0$, we have $\widetilde{\sigma}_n \in \mathcal{A}\sigma_n$.*

Proof. If we show that

$$\varphi(\widetilde{\sigma}_n) = (-1)^n v^{-n^2+n}\{H\}_n \sum_{i=n}^{2n}(-1)^i\{i\}_n \begin{bmatrix} 2n+1 \\ i+1 \end{bmatrix}\{H+n-i\}_n, \tag{10.9}$$

then

$$\varphi(\widetilde{\sigma}_n) \in \mathrm{Span}_{\mathcal{A}}\{K^{2n}, K^{2n-2}, \dots, K^{-2n}\}.$$

By Lemma 10.4, $\widetilde{\sigma}_n \in \mathrm{Span}_{\mathcal{A}}\{1, C^2, C^4, \dots, C^{2n}\}$, and the statement follows.

Let us now prove (10.9). It follows from (10.8) and the fact that $t \triangleright x = \epsilon(t)x$ for $t \in \mathcal{U}^0$, $x \in \Gamma_0\mathcal{U}$, that

$$\widetilde{\sigma}_n = \sum_{i=0}^{2n}(-1)^i\{2n+1\}_{2n-i}F^{(i)}e^i \triangleright F^{(n)}K^{-n}e^n. \tag{10.10}$$

If $i < n$, then $\varphi(F^{(i)}e^i \triangleright F^{(n)}K^{-n}e^n) = 0$. Otherwise, we can prove, using (10.6) and (10.7), that

$$\begin{aligned}\varphi(F^{(i)}e^i \triangleright F^{(n)}K^{-n}e^n) &= (-1)^i v^{-i(i-1)+2ni-2n^2}\begin{bmatrix} i \\ n \end{bmatrix}\{H\}_n\{H+n-i\}_n K^{i-n} \\ &\quad\times \sum_{j=0}^{i-n}(-1)^j v^{j(i-1)-2nj}\begin{bmatrix} i-n \\ j \end{bmatrix}\{H-n\}_{i-n-j}\{H+n+j-i\}_j.\end{aligned}$$

By Lemma 10.2, the sum in the right hand side is equal to

$$(-1)^{i-n}v^{(i-n)^2-(i-n)}\{n\}_{i-n}K^{-i+n}.$$

Then (10.9) easily follows. $\square$

Proposition 10.6. *For $n \geq 0$, we have $\widetilde{\sigma}_n = v^{-n^2+n}\{n\}!\sigma_n$.*

Proof. For an element $t \in \mathcal{U}^0 = \mathcal{A}[K, K^{-1}]$ and $p \in \mathbb{Z}$, let $c_p(t) \in \mathcal{A}$ denote the coefficient of K^p in t. Note that $c_n(\{H+r\}_n) = v^{\sum_{s=r-n+1}^{r} s} = v^{n(2r-n+1)/2}$ for $r \in \mathbb{Z}$. Hence

$$c_{2n}(\varphi(\sigma_n)) = c_{2n}(\{H\}_n\{H+n+1\}_n) = c_n(\{H\}_n)c_n(\{H+n+1\}) = v^{2n}.$$

Using (10.9), we obtain

$$c_{2n}(\varphi(\widetilde{\sigma}_n)) = (-1)^n v^{-n^2+2n}\{n\}!\sum_{i=n}^{2n}(-1)^i v^{-ni}\begin{bmatrix} i \\ n \end{bmatrix}\begin{bmatrix} 2n+1 \\ i+1 \end{bmatrix}.$$

By Lemma 10.1,

$$c_{2n}(\varphi(\widetilde{\sigma}_n)) = (-1)^n v^{-n^2+2n}\{n\}!(-1)^n v^n = v^{-n^2+3n}\{n\}!.$$

Hence, by Lemma 10.5,

$$\widetilde{\sigma}_n = \frac{c_{2n}(\varphi(\widetilde{\sigma}_n))}{c_{2n}(\varphi(\sigma_n))}\sigma_n = \frac{v^{-n^2+3n}\{n\}!}{v^{2n}}\sigma_n = v^{-n^2+n}\{n\}!\sigma_n. \quad \square$$

Lemma 10.7. *For $n \geq 0$,*

$$\widetilde{\sigma}_n = \sum_{j=0}^{2n}(-1)^j v^{j(-j-1+2n)+2n}\{2n-j\}!(e^j \triangleright F^{(n)})(F^{(j)} \triangleright K^{-n}e^n).$$

Proof. Since $e^i F^{(i)} - F^{(i)} e^i \in \mathcal{U}\{H\}$ by (3.1), we have $F^{(i)} e^i \triangleright x = e^i F^{(i)} \triangleright x$ for $x \in \Gamma_0\mathcal{U}$. Hence it follows from (3.1), (3.6) and (3.8), and $e \triangleright K^{-n} e^n = F^{(p)} \triangleright F^{(r)} = 0$ ($p \geq 1$) that

$$\begin{aligned} F^{(i)} e^i \triangleright F^{(n)} K^{-n} e^n &= e^i F^{(i)} \triangleright F^{(n)} K^{-n} e^n \\ &= e^i \triangleright F^{(n)} (F^{(i)} \triangleright K^{-n} e^n) \\ &= \sum_{j=0}^{i} v^{-j(i-j)} \begin{bmatrix} i \\ j \end{bmatrix} (K^{i-j} e^j \triangleright F^{(n)})(e^{i-j} F^{(i)} \triangleright K^{-n} e^n). \end{aligned}$$

Since

$$K^{i-j} e^j \triangleright F^{(n)} = v^{2(i-j)(j-n)} e^j \triangleright F^{(n)}$$

and

$$e^{i-j} F^{(i)} \triangleright K^{-n} e^n = F^{(j)} \{H - j\}_{i-j} \triangleright K^{-n} e^n = F^{(j)} \{2n - j\}_{i-j} \triangleright K^{-n} e^n,$$

we have

$$F^{(i)} e^i \triangleright F^{(n)} K^{-n} e^n = \sum_{j=0}^{i} v^{(i-j)(j-2n)} \begin{bmatrix} i \\ j \end{bmatrix} \{2n - j\}_{i-j} (e^j \triangleright F^{(n)})(F^{(j)} \triangleright K^{-n} e^n).$$

By (10.10),

$$\begin{aligned} \widetilde{\sigma}_n &= \sum_{i=0}^{2n} (-1)^i \{2n+1\}_{2n-i} \sum_{j=0}^{i} v^{(i-j)(j-2n)} \begin{bmatrix} i \\ j \end{bmatrix} \{2n - j\}_{i-j} (e^j \triangleright F^{(n)})(F^{(j)} \triangleright K^{-n} e^n) \\ &= \sum_{j=0}^{2n} v^{-j(j-2n)} \{2n - j\}! \left(\sum_{i=j}^{2n} (-1)^i v^{i(j-2n)} \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} 2n+1 \\ i+1 \end{bmatrix} \right) (e^j \triangleright F^{(n)})(F^{(j)} \triangleright K^{-n} e^n). \end{aligned}$$

Here we used the identity $\{2n-j\}_{i-j}\{2n+1\}_{2n-i} = \{2n-j\}! \begin{bmatrix} 2n+1 \\ i+1 \end{bmatrix}$. By Lemma 10.1, the sum over i is equal to $(-1)^j v^{2n-j}$. Hence the statement follows. $\square$

Proof of Proposition 9.4. We have to show that $\widetilde{\sigma}_n \in \{n\}!\mathcal{U}_n^e$. In view of Lemma 10.7 and $F^{(j)} \triangleright K^{-n} e^n \in \mathcal{U}_n^e$, it suffices to prove that

$$\{2n - j\}!(e^j \triangleright F^{(n)}) \in \{n\}!\mathcal{U}$$

for each $j = 0, \ldots, 2n$. Note that the case $0 \leq j \leq n$ is trivial. We assume $n < j \leq 2n$ in the following. In view of Lemma 10.3, it suffices to show that if $0 \leq l \leq k \leq n < j \leq 2n$, then

$$\{2n - j\}! \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} \{n - k + l\}_{j-k} \in \{n\}!\mathcal{A}. \tag{10.11}$$

Note that $\{n - k + l\}_{j-k} = 0$ unless $l - j + n \geq 0$. If $l - j + n \geq 0$, then (10.11) is equal to

$$\{n\}!\{n - k\}!\theta(j - n, k - l, l - j + n, n - k),$$

which is contained in $\{n\}!\{n - k\}!\mathcal{A} \subset \{n\}!\mathcal{A}$ by Lemma 10.8 below. This completes the proof. $\square$

10.4. An integrality lemma

For $a, b, x, y \geq 0$, we set

$$\theta(a, b, x, y) = \psi(a, x, y) \begin{bmatrix} x + y \\ x \end{bmatrix} \begin{bmatrix} b + x + y \\ b \end{bmatrix} \begin{bmatrix} 2a + b + x + y \\ a \end{bmatrix} \in \mathbb{Q}(v)$$

where

$$\psi(a, x, y) = \frac{\{a + x + y\}!\{a\}!}{\{a + x\}!\{a + y\}!} \in \mathbb{Q}(v).$$

Lemma 10.8. *If* $a, b, x, y \geq 0$, *then* $\theta(a, b, x, y) \in \mathcal{A}$.

Proof. For each $n \geq 1$, let $\phi_n \in \mathcal{A}$ denote the "balanced cyclotomic polynomial"

$$\phi_n = v^{-\deg \Phi_n(q)} \Phi_n(q),$$

where $\Phi_n(q) \in \mathbb{Z}[q]$ is the nth cyclotomic polynomial. We have

$$\{n\}! = \prod_{d \geq 1} \phi_d^{\lfloor \frac{n}{d} \rfloor}, \quad \begin{bmatrix} m+n \\ m \end{bmatrix} = \prod_{d \geq 1} \phi_d^{\lfloor \frac{m+n}{d} \rfloor - \lfloor \frac{m}{d} \rfloor - \lfloor \frac{n}{d} \rfloor}$$

for $m, n \geq 0$. Here $\lfloor r \rfloor = \max\{i \in \mathbb{Z} \mid i \leq r\}$ for $r \in \mathbb{Q}$. Set for $a, b, x, y \geq 0$

$$\theta_d(a, b, x, y) = \psi_d(a, x, y) + \tau_d(x, y) + \tau_d(b, x+y) + \tau_d(a, a+b+x+y),$$

where

$$\psi_d(a, x, y) = \left\lfloor \frac{a+x+y}{d} \right\rfloor + \left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{a+x}{d} \right\rfloor - \left\lfloor \frac{a+y}{d} \right\rfloor,$$

$$\tau_d(x, y) = \left\lfloor \frac{x+y}{d} \right\rfloor - \left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{y}{d} \right\rfloor.$$

We have

$$\theta(a, b, x, y) = \prod_{d \geq 1} \phi_d^{\theta_d(a,b,x,y)}.$$

It suffices to prove that if $d \geq 1$ and $a, b, x, y \geq 0$, then $\theta_d(a, b, x, y) \geq 0$. Suppose by contradiction that there are integers $d \geq 1$, $a, b, x, y \geq 0$ such that $\theta_d(a, b, x, y) < 0$. Since $\tau_d(r, s) \geq 0$ for all $r, s \geq 0$, we have

$$\psi_d(a, x, y) < 0. \tag{10.12}$$

Note that if $a', b', x', y' \geq 0$ are congruent mod d to a, b, c, d, respectively, then

$$\theta_d(a', b', x', y') = \theta_d(a, b, x, y), \;\; \psi_d(a', x', y') = \psi_d(a, x, y), \;\; \tau_d(x', y') = \tau_d(x, y).$$

For $n \in \mathbb{Z}$, let $\widetilde{n}$ denote the unique integer $m \in \{0, 1, \ldots, d-1\}$ such that $m \equiv n (\mathrm{mod}\ d)$. For an inequality P, let $[P]$ stand for 1 if P holds and 0 otherwise. We have

$$\begin{aligned} \psi_d(a, x, y) &= \psi_d(\widetilde{a}, \widetilde{x}, \widetilde{y}) = \left\lfloor \frac{\widetilde{a}+\widetilde{x}+\widetilde{y}}{d} \right\rfloor + \left\lfloor \frac{\widetilde{a}}{d} \right\rfloor - \left\lfloor \frac{\widetilde{a}+\widetilde{x}}{d} \right\rfloor - \left\lfloor \frac{\widetilde{a}+\widetilde{y}}{d} \right\rfloor \\ &= [d \leq \widetilde{a}+\widetilde{x}+\widetilde{y}] + [2d \leq \widetilde{a}+\widetilde{x}+\widetilde{y}] - [d \leq \widetilde{a}+\widetilde{x}] - [d \leq \widetilde{a}+\widetilde{y}]. \end{aligned} \tag{10.13}$$

Since either $d \leq \widetilde{a}+\widetilde{x}$ or $d \leq \widetilde{a}+\widetilde{y}$ implies $d \leq \widetilde{a}+\widetilde{x}+\widetilde{y}$, it follows from (10.12) and (10.13) that

$$\psi_d(a, x, y) = -1, \tag{10.14}$$

$$d \leq \widetilde{a}+\widetilde{x}, \;\; d \leq \widetilde{a}+\widetilde{y}, \;\; \widetilde{a}+\widetilde{x}+\widetilde{y} < 2d. \tag{10.15}$$

By (10.14) and the assumption $\theta_d(a, b, x, y) < 0$, we have

$$\tau_d(x, y) = 0, \tag{10.16}$$

$$\tau_d(b, x+y) = 0, \tag{10.17}$$

$$\tau_d(a, a+b+x+y) = 0. \tag{10.18}$$

By (10.16), we have $0 = \tau_d(x, y) = \tau_d(\widetilde{x}, \widetilde{y}) = [d \leq \widetilde{x}+\widetilde{y}]$, and hence

$$\widetilde{x}+\widetilde{y} < d. \tag{10.19}$$

Set $z = x + y$. In view of (10.19), we have $\widetilde{z} = \widetilde{x}+\widetilde{y}$. By (10.15), we have $2d - 2\widetilde{a} \leq \widetilde{x}+\widetilde{y} < d$, and hence

$$2d - 2\widetilde{a} \leq \widetilde{z}. \tag{10.20}$$

By (10.17), we have $0 = \tau_d(b,z) = \tau_d(\widetilde{b},\widetilde{z}) = [d \le \widetilde{b}+\widetilde{z}]$, and hence

$$\widetilde{b}+\widetilde{z} < d. \tag{10.21}$$

Set $w = b+z = b+x+y$. By (10.21), we have $\widetilde{w} = \widetilde{b+z} = \widetilde{b}+\widetilde{z}$. By (10.20), we have $2d-2\widetilde{a} \le \widetilde{z} \le \widetilde{z}+\widetilde{b} = \widetilde{w}$, and hence $2d \le 2\widetilde{a}+\widetilde{w}$. Therefore

$$\tau_d(a,a+w) = \tau_d(\widetilde{a},\widetilde{a}+\widetilde{w}) = \left\lfloor \frac{2\widetilde{a}+\widetilde{w}}{d} \right\rfloor - \left\lfloor \frac{\widetilde{a}+\widetilde{w}}{d} \right\rfloor - \left\lfloor \frac{\widetilde{a}}{d} \right\rfloor = 2 - \left\lfloor \frac{\widetilde{a}+\widetilde{w}}{d} \right\rfloor - 0 \ge 1,$$

which contradicts (10.18). This completes the proof. □

11. $\mathcal{A}_q$-forms

The $\mathbb{Q}(v)$-algebra U admits a "$\mathbb{Q}(q)$-form" U_q, which is the $\mathbb{Q}(q)$-subalgebra of U generated by the elements K, K^{-1}, vE, and F, and inherits from U a Hopf $\mathbb{Q}(q)$-algebra structure. We have the $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathbb{Q}(q)$-algebra structure $U = U_q \oplus vU_q$. We will call this $(\mathbb{Z}/2\mathbb{Z})$-grading the *v-grading*.

The $\mathcal{A}$-subalgebras $U_{\mathcal{A}}$, $\mathcal{U}$, and $\overline{U}$ of U are homogeneous in the v-grading, and hence admit $\mathcal{A}_q$-forms $U_{\mathcal{A}_q} = U_{\mathcal{A}} \cap U_q$, $\mathcal{U}_q = \mathcal{U} \cap U_q$, and $\overline{U}_q = \overline{U} \cap U_q$, respectively. In particular, $\mathcal{U}_q$ is the $\mathcal{A}_q$-subalgebra of $\mathcal{U}$ generated by the elements K, K^{-1}, e, and the $v^{-\frac{1}{2}n(n-1)}F^{(n)}$ for $n \ge 1$, and inherits from $\mathcal{U}$ a Hopf $\mathcal{A}_q$-algebra structure. The 0th graded part $\mathcal{G}_0\mathcal{U}_q$ of $\mathcal{U}_q$ is generated by the elements K^2, K^{-2}, e, and the $v^{-\frac{1}{2}n(n-1)}F^{(n)}K^n$ for $n \ge 1$.

Since the ideals $\mathcal{U}_n$ are homogeneous in the v-grading, the algebra $\hat{\mathcal{U}}$ has a $(\mathbb{Z}/2\mathbb{Z})$-graded algebra structure $\hat{\mathcal{U}} = \hat{\mathcal{U}}_q \oplus v\hat{\mathcal{U}}_q$ with $\hat{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/(\mathcal{U}_n \cap \mathcal{U}_q)$. Similarly, $\dot{\mathcal{U}}, \check{\mathcal{U}}$, and $\tilde{\mathcal{U}}$ is $(\mathbb{Z}/2\mathbb{Z})$-graded with the 0th graded part being

$$\dot{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/(\mathcal{U}_1 \cap \mathcal{U}_q)^n, \quad \check{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/\mathcal{U}_q e^n \mathcal{U}_q, \quad \tilde{\mathcal{U}}_q = \mathrm{Im}(\check{\mathcal{U}}_q \to \hat{\mathcal{U}}_q).$$

Since the ideals $\mathcal{U}_n$, $(\mathcal{U}_1)^n$, and $\mathcal{U}_n^e$ $(n \ge 0)$ are homogeneous in the v-grading, the algebra $\hat{\mathcal{U}}$ (resp. $\dot{\mathcal{U}}, \check{\mathcal{U}}, \tilde{\mathcal{U}}$) has an $\hat{\mathcal{A}}_q$- (resp. $\dot{\mathcal{A}}_q$-, $\mathcal{A}_q$-, $\mathcal{A}_q$-)form, which we will denote by $\hat{\mathcal{U}}_q$ (resp. $\dot{\mathcal{U}}_q, \check{\mathcal{U}}_q, \tilde{\mathcal{U}}_q$). These algebras can be regarded as completions of $\mathcal{U}_q$:

$$\hat{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/(\mathcal{U}_n \cap \mathcal{U}_q), \quad \dot{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/(\mathcal{U}_1 \cap \mathcal{U}_q)^n, \quad \check{\mathcal{U}}_q = \varprojlim_n \mathcal{U}_q/\mathcal{U}_q e^n \mathcal{U}_q, \quad \tilde{\mathcal{U}}_q = \mathrm{Im}(\check{\mathcal{U}}_q \to \hat{\mathcal{U}}_q).$$

Remark 11.1. Each term of the quasi-R-matrix (1.2) is contained in $\mathcal{U}_q \otimes_{\mathcal{A}_q} \mathcal{U}_q \subset \mathcal{U} \otimes_{\mathcal{A}} \mathcal{U}$. (Note that $\mathcal{U} \otimes_{\mathcal{A}} \mathcal{U} = (\mathcal{U}_q \otimes_{\mathcal{A}_q} \mathcal{U}_q) \oplus v(\mathcal{U}_q \otimes_{\mathcal{A}_q} \mathcal{U}_q)$.) Therefore we can regard Θ as an element of the completed tensor product $\check{\mathcal{U}}_q \check{\otimes} \check{\mathcal{U}}_q \subset \check{\mathcal{U}} \check{\otimes} \check{\mathcal{U}}$ of two copies of $\check{\mathcal{U}}_q$, and hence as an element of the image $\tilde{\mathcal{U}}_q \tilde{\otimes} \tilde{\mathcal{U}}_q = \mathrm{Im}(\check{\mathcal{U}}_q \check{\otimes} \check{\mathcal{U}}_q \to \hat{\mathcal{U}}_q \hat{\otimes} \hat{\mathcal{U}}_q)$, where $\hat{\mathcal{U}}_q \hat{\otimes} \hat{\mathcal{U}}_q$ is the completed tensor product of two copies of $\hat{\mathcal{U}}_q$.

The centers of $\mathcal{U}, \hat{\mathcal{U}}, \dot{\mathcal{U}}$, and $\tilde{\mathcal{U}}$ are homogeneous in the v-grading, and Theorems 9.2, 9.9 and 9.13–9.15 hold for $\mathcal{U}_q$ and its completions. In particular, the following theorem, which we use in [5], holds.

Theorem 11.2. *We have*

$$Z(\tilde{\mathcal{U}}_q) \simeq \varprojlim_n \mathcal{A}_q[vC]/(\sigma_n),$$
$$Z(\mathcal{G}_0\tilde{\mathcal{U}}_q) = \mathcal{G}_0 Z(\tilde{\mathcal{U}}_q) \simeq \varprojlim_n \mathcal{A}_q[C^2]/(\sigma_n).$$

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